

REFINED SCATTERING AND HERMITIAN SPECTRAL THEORY FOR LINEAR HIGHER-ORDER SCHRÖDINGER EQUATIONS

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ABSTRACT. The Cauchy problem for a linear $2m$ th-order Schrödinger equation

$$(0.1) \quad u_t = -i(-\Delta)^m u \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad u|_{t=0} = u_0; \quad m \geq 1 \text{ is an integer,}$$

is studied, for initial data u_0 in the weighted space $L_{\rho^*}^2(\mathbb{R}^N)$, with $\rho^*(x) = e^{|x|^\alpha}$ and $\alpha = \frac{2m}{2m-1} \in (1, 2]$. The following **five** problems are studied:

(I) A sharp asymptotic behaviour of solutions as $t \rightarrow +\infty$ is governed by a discrete spectrum and a countable set Φ of the eigenfunctions of the linear rescaled operator

$$\mathbf{B} = -i(-\Delta)^m + \frac{1}{2m} y \cdot \nabla + \frac{N}{2m} I, \quad \text{with the spectrum } \sigma(\mathbf{B}) = \{\lambda_\beta = -\frac{|\beta|}{2m}, |\beta| \geq 0\}.$$

(II) Finite-time blow-up local structures of nodal sets of solutions as $t \rightarrow 0^-$ and formation of “multiple zeros” are described by the eigenfunctions being *generalized Hermite polynomials*, of the “adjoint” operator

$$\mathbf{B}^* = -i(-\Delta)^m - \frac{1}{2m} y \cdot \nabla, \quad \text{with the same spectrum } \sigma(\mathbf{B}^*) = \sigma(\mathbf{B}).$$

Applications of these spectral results also include: (III) a unique continuation theorem, and (IV) boundary characteristic point regularity issues.

Some applications are discussed for more general linear PDEs and for the nonlinear Schrödinger equations in the focusing (“+”) and defocusing (“−”) cases

$$u_t = -i(-\Delta)^m u \pm i|u|^{p-1}u \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad \text{where } p > 1,$$

as well as for (V) the quasilinear Schrödinger equation of a “porous medium type”

$$u_t = -i(-\Delta)^m(|u|^n u) \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad \text{where } n > 0.$$

For the latter one, the main idea towards countable families of *nonlinear eigenfunctions* is to perform a homotopic path $n \rightarrow 0^+$ and to use spectral theory of the pair $\{\mathbf{B}, \mathbf{B}^*\}$.

1. INTRODUCTION: DUALITY OF GLOBAL AND BLOW-UP SCALINGS, HERMITIAN SPECTRAL THEORY, AND REFINED SCATTERING

1.1. Basic Schrödinger equations and key references. Consider the linear $2m$ th-order Schrödinger equation (the LSE- $2m$), with any integer $m \geq 1$,

$$(1.1) \quad u_t = -i(-\Delta)^m u \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad u|_{t=0} = u_0,$$

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where Δ is the Laplace operator in \mathbb{R}^N , for initial data u_0 in some weighted L^2 -space, to be introduced. Here $m = 1$ corresponds to the classic Schrödinger equation

$$(1.2) \quad i u_t = -\Delta u \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+,$$

which very actively entered general PDE theory from Quantum Mechanics in the 1920s.

It is not an exaggeration to say that, nowadays, linear and nonlinear Schrödinger type equations are the most popular PDE models of modern mathematics among other types of equations. In Appendix A, we present a “mathematical evidence” for that by using simple data from the **MathSciNet**. It not possible to express how deep is mathematical theory developed for models such as (1.2), (1.1), and related semilinear ones. We refer to well-known monographs [86, 10], which cover classes of both linear and nonlinear PDEs.

Concerning the results that are more closely related to the subject of this paper, we note that scattering L^2 and $L_{x,t}^{q,r}$ theory for (1.2) has been fully developed in the works by Stein, Tomas, Segal, Strichartz in the 1970s with later further involved estimates in more general spaces by Ginibre and Velo, Yajima, Cazenave and Weissler, Montgomery-Smith, Keel, Tao, and many others; see [47] and [91] for references concerning these, as well as optimal $L_{x,t}^{q,r}$ estimates for the non-homogeneous equation

$$(1.3) \quad i u_t = -\Delta u + F(x, t) \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R},$$

as well as more recent papers [50, 82, 99].

The $2m$ th-order counterpart (1.1) was also under scrutiny for a long period. We refer to Ablowitz–Segur’s monograph [1], Ivano–Kosevich [43], Turitsyn [89], Karpman [44], and Karpman–Shagalov [45] for physical, symmetry, and other backgrounds of higher-order Schrödinger-type semilinear models (see also [100] for extra motivations from nonlinear optics), [74] for first existence and uniqueness results, and more recent papers [5, 6, 13, 33, 35, 36, 68, 72, 73, 101] as an account for further applied and rigorous research, as well as other earlier key references and survey, in this fundamental area of modern PDE theory.

1.2. Discrete real spectra, “Hermitian spectral history”, and our motivation.

Actually, the developed here refined scattering theory is rather general, so our methods can be applied not only to the Schrödinger equations such as (1.1), but also to practically any linear evolution PDEs with constant or perturbed smooth coefficients and classic solutions. Formulating the approach rather loosely, we claim that, for (1.1), the most principal part is played by the spectral theory for the following rescaled operator:

$$(1.4) \quad \mathbf{B}^* = -i(-\Delta_y)^m - \frac{1}{2m} y \cdot \nabla_y, \quad \text{where} \quad y = \frac{x}{(-t)^{1/2m}} \quad \text{for} \quad t < 0$$

is the corresponding *Sturmian blow-up backward spatial variable* at the focusing point $(x, t) = (0, 0^-)$. In 1836, C. Sturm used the backward variable $y = \frac{x}{\sqrt{-t}}$, $t \rightarrow 0^-$, for the heat equation with a potential,

$$(1.5) \quad u_t = u_{xx} + q(x)u,$$

in his seminal paper [85] [The pioneering work [85] was practically fully forgotten for about 150 years!—practically until the 1980s,—plausibly the most amazing and striking

such an example in the whole history of mathematics ever], where he formulated his two fundamental theorems on zeros sets of solutions $u = u(x, t)$ of (1.5). This remarkable history, with many extensions about, can be found in [23, Ch. 1] with precise statements of Sturm's results of the 1836 written in his original notations.

The operator (1.4) and its adjoint below are then respectively defined in weighted L^2 -spaces, with a special “radiation conditions at infinity” to be specified. More than half of the paper is devoted to the study of (1.4) and its “adjoint” operator

$$(1.6) \quad \mathbf{B} = -i(-\Delta_y)^m + \frac{1}{2m} y \cdot \nabla_y + \frac{N}{2m} I, \quad y = \frac{x}{t^{1/2m}}, \quad t > 0 \quad (\text{the forward variable}).$$

Indeed, (1.4) is a perturbation of the original one in (1.1). Though the perturbation is of the first order, the coefficient y therein is unbounded as $y \rightarrow +\infty$, so this changes the natural space $L^2(\mathbb{R}^N)$ and moves the operator into an essentially weighted metric. An amazing property of (1.4) is that, being properly defined, it has the discrete spectrum

$$(1.7) \quad \sigma(\mathbf{B}^*) = \left\{ \lambda_\beta = -\frac{|\beta|}{2m}, \quad |\beta| = 0, 1, 2, \dots \right\} \quad (\beta \text{ is a multiindex in } \mathbb{R}^N),$$

and all the eigenfunctions $\{\psi_\beta^*(y)\}$ are finite polynomials (*generalized Hermite* ones).

These properties directly match the classic results for the *heat equation* for $m = 1$:

$$(1.8) \quad \begin{aligned} u_t = u_{xx} &\implies \mathbf{B}^* = D_y^2 - \frac{1}{2} y D_y, \quad \text{with } \sigma(\mathbf{B}^*) = \left\{ \lambda_\beta = -\frac{l}{2}, \quad l \geq 0 \right\}, \\ \text{so } (\mathbf{B}^*)^* = \mathbf{B}^* &\text{ in } L_{\hat{\rho}^*}^2(\mathbb{R}), \quad \hat{\rho}^*(y) = e^{-\frac{y^2}{4}}, \quad \mathbf{B} = D_y^2 + \frac{1}{2} y D_y + \frac{1}{2} I, \end{aligned}$$

where \mathbf{B} is defined in the adjoint space $L_{\hat{\rho}}^2$, with $\hat{\rho} = \frac{1}{\hat{\rho}^*}$, and $(\mathbf{B})^* = \mathbf{B}^*$ in the dual metric of L^2 , etc. In modern language, for (1.8), the spectrum (1.7) with $m = 1$ and the classic Hermite polynomials (introduced in detail about 1870) as eigenfunctions were already constructed by C. Sturm in 1836 [85]. As we mentioned already, this led Sturm to formulate his two fundamental theorems on the structure of multiple zeros of solutions of parabolic equations and on nonincrease in time of the zero number (or sign changes of solutions); we refer again to [23, Ch. 1] for a full history and key further references and extensions. The operator \mathbf{B}^* in (1.8), admitting a natural N -dimensional extension similar to (1.4), remains one of the key objectives in general theory of linear self-adjoint operators; see Birman–Solomjak’s monograph [9].

The spectral results for (1.8) and their consequences for the asymptotic behaviour for second-order parabolic equations are classic and well-known since the 1830s, with further extensions as orthonormal polynomial families by Hermite himself from the 1870s.

However, and this looks like a truly amazing fact, a direct extension of such classic results to other classes of PDEs took a lot of time. For instance, similar spectral theory for the 1D *bi-harmonic equation* (cf. with the one in the first line in (1.8))

$$(1.9) \quad u_t = -u_{xxxx} \implies \mathbf{B}^* = -D_y^4 - \frac{1}{4} y D_y, \quad \text{with } \sigma(\mathbf{B}^*) = \left\{ \lambda_\beta = -\frac{l}{4}, \quad l = 0, 1, 2, \dots \right\},$$

etc., was developed in 2004 [17], i.e., 168 years later after Sturm’s pioneering discovery for $m = 1$ in 1836! As a certain (but seems not that convincing) excuse, note that the operator \mathbf{B}^* in (1.9) *is not self-adjoint* in no weighted space, though keeps having discrete

real spectrum, polynomial eigenfunctions (naturally called *generalized Hermite ones*), and a number of other nice and typical from self-adjoint theory properties.

Therefore, our goal is to show that similar issues remain true for our rescaled Schrödinger operators (1.4) and (1.6) (so we are talking about a *spectral pair* $\{\mathbf{B}, \mathbf{B}^*\}$ of non-self-adjoint ones), which clearly have analogous structures, though the mathematics becomes essentially more involved than for (1.9), to say nothing of the well-studied self-adjoint case (1.8).

1.3. Layout of the paper: duality of global and blow-up asymptotics. In Section 2, we describe some properties of the *fundamental solution* of (1.1) given by

$$(1.10) \quad b(x, t) = t^{-\frac{N}{2m}} F(y), \quad y = \frac{x}{t^{1/2m}} \implies \mathbf{B}F = 0 \quad \text{in } \mathbb{R}^N.$$

Next sections are mainly devoted to the following two asymptotic problems for (1.1):

Application I: GLOBAL ASYMPTOTICS AS $t \rightarrow +\infty$, SECTIONS 3 AND 5. The asymptotic behaviour as $t \rightarrow +\infty$ of solutions is governed by the eigenfunctions of (1.6):

$$(1.11) \quad \mathbf{B} = -i(-\Delta)^m + \frac{1}{2m} y \cdot \nabla + \frac{N}{2m} I, \quad \text{with } \sigma(\mathbf{B}) = \{\lambda_\beta = -\frac{|\beta|}{2m}, |\beta| \geq 0\},$$

which demands a proper definition of its domain by a spectral decomposition (Section 3) and by a traditional spectral theory involving careful using poles of the resolvent, non-classic “radiation conditions” at infinity (see an alternative approach in [26]), etc. More precisely, we establish that the discrete spectrum and the eigenfunction set for the operator (1.11) describe all the possible asymptotics as $t \rightarrow +\infty$ of solutions of (1.1) for any data $u_0 \in L^2_{\rho^*}(\mathbb{R}^N)$. The exponential weights (recall the change $\rho \mapsto \rho^*$ relative to (1.8) done by some clear and natural reasons)

$$(1.12) \quad \rho^*(y) = e^{|y|^\alpha}, \quad \text{with } \alpha = \frac{2m}{2m-1}, \quad \text{and } \rho(y) = \frac{1}{\rho^*(y)} = e^{-|y|^\alpha},$$

are properly introduced in Section 3. It is curious that, even in the classic case $m = 1$, i.e., for (1.2), we were not able to find any essential traces of such a full refined scattering theory (except some particular results often admitting not-that-clear interpretation) and corresponding spectral properties in the vast existing scattering literature¹.

The classic real analogy and forerunner of (1.11) is the self-adjoint operator for $m = 1$ (we apologize for the necessary change of the weights here, $\rho \mapsto \rho^*$; cf. (1.12))

$$(1.13) \quad \mathbf{B} = \Delta + \frac{1}{2} y \cdot \nabla + \frac{N}{2} I \equiv \frac{1}{\rho} \nabla \cdot (\rho \nabla) + \frac{N}{2} I \quad \text{in } L^2_\rho(\mathbb{R}^N), \quad \text{with } \rho = e^{\frac{|y|^2}{4}}.$$

¹The authors do not still believe that optimal and sharp large-time ($t \rightarrow \infty$) asymptotic theory for the classic LSE (1.2) in \mathbb{R}^N has not been developed in *full details* (some particular results have been indeed known seems) since this PDE burst into quantum mechanics and mathematical physics in the 1920s. As was mentioned, a similar (and even stronger) asymptotic theory for the 1D heat equation $u_t = \Delta u$ (cf. (1.1) for $m = 1$) is known since Sturm’s analysis [85] of formation of multiple zeros of solutions (we will do this for (1.1) in Section 6) obtained in 1836! The authors will be very pleased to get rid of such a quite surprising delusion, but also would be naturally satisfied to know that this, though again unbelievably, is done in full details for the first time in the present paper.

As we have pointed out already, its real discrete spectrum $\sigma(\mathbf{B}) = \{-\frac{l}{2}, l \geq 0\}$ and eigenfunctions as *Hermite polynomials*, multiplied by the Gaussian, are known from, at least, the 1830s, and are associated with the names of Sturm and Hermite; see [23, § 1.2] for more history and original Sturm's calculations, and [9, p. 48] for a fuller account of applications of these separable polynomials in self-adjoint linear operator theory.

Thus, we are obliged here to develop Hermitian-like spectral theory for the $2m$ th-order rescaled Schrödinger operator (1.11), and this is an unavoidable task if we want to reach an optimal classification of large time behaviour for the non-stationary LSE (1.1).

Application II: BLOW-UP ASYMPTOTICS AS $t \rightarrow 0^-$, SECTIONS 4, 6, APPENDICES B AND D. Alternatively, for data $u_0 \in L^2_{\rho^*}(\mathbb{R}^N)$, using blow-up scaling at a finite point as $x \rightarrow 0$ and $t \rightarrow T^- = 0^-$, we show that this behaviour of solutions and local structure of their nodal sets are described by the eigenfunctions of the linear operator (1.4), which is “adjoint” to \mathbf{B} in a sense (but not in the standard dual L^2 -metric; an *indefinite metric* should be involved, which will be carefully explained). The discrete real spectrum (1.7) remains the same as in (1.11).

A key point is that the eigenfunctions of \mathbf{B}^* are *generalized Hermite polynomials* $\{\psi^*_\beta(y)\}$, so that the nodal sets of solutions of (1.1) are locally governed by zero surfaces generated by these polynomials *only* (there exists a countable, complete, and closed set of those). **Application III:** This allows us to state in Section 6 a sharp *uniqueness continuation* theorem for (1.1): if $\text{Im } u(x, t)$ (or $\text{Re } u(x, t)$) has, roughly speaking,

(1.14) a rescaled nodal set component not decomposable into a finite combination
from polynomial surfaces $\{\text{Im } \psi^*_\beta(y) = 0, |\beta| \geq 0\}$, then $u(x, t) \equiv 0$.

Some further applications of these spectral results are also discussed.

In Sections 5 and 6, possible applications of Hermitian spectral theory are studied for more general linear PDEs and for the $2m$ th-order nonlinear Schrödinger equation (the NLSE- $2m$)

$$(1.15) \quad u_t = -i(-\Delta)^m u \pm i|u|^{p-1}u \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad \text{where } p > 1$$

and the sign “+” corresponds to the focusing (blow-up) model, while “−” gives a defocusing one. See [48, 66, 67, 77, 81, 93] as a guide concerning the modern research of both semilinear PDEs (1.15).

Application IV: In Appendix B, we show how to apply the spectral results to the classic problem of the *regularity of a boundary characteristic point* for the linear Schrödinger equation. It seems that, in the present setting of rather arbitrary “backward paraboloids” at characteristic points, such issues were not addressed in the existing literature.

(ii) **Application V: QUASILINEAR SCHRÖDINGER EQUATION, APPENDIX C.** This is a most “risky” application of our refined scattering spectral theory for the spectral pair $\{\mathbf{B}, \mathbf{B}^*\}$, so we also put this into an appendix². Thus, exhibiting a certain necessary

²This and some other non-entirely-rigorous parts of our applications well resonate with two Tao's comments in his “What is good mathematics?”: (i) “... mathematical rigour, while highly important, is only one component of what determines a quality piece of mathematics” [87, p. 624], and (ii) “... we

bravery, we develop some basics of a “nonlinear eigenfunction theory” for a *quasilinear Schrödinger equation* (the QLSE) of the form

$$(1.16) \quad u_t = -i(-\Delta)^m(|u|^n u) \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad \text{where } n > 0$$

is a fixed parameter. The applications and some history/references concerning such rather unusual quasilinear PDEs are explained therein. Here, we intend to reconstruct a proper connection between linear and “nonlinear” spectral theory by performing a continuity homotopic path $n \rightarrow 0^+$, which establishes a link between “nonlinear eigenvalue problems” for (1.16) and the linear one developed for (1.1). As a result, we predict existence of a countable family of the so-called n -branches of solutions, which are originated at $n = 0$ from eigenfunctions on the corresponding eigenspaces for the linear spectral pair $\{\mathbf{B}, \mathbf{B}^*\}$.

We strongly believe that the results of Hermitian spectral theory developed can be useful for attacking a number of open problems concerning blow-up behaviour for (1.15), (1.16), and others. We plan to explain this in a forthcoming paper. Meantime, we just comment on that standard blow-up rescaling (*q.v.* (4.6)) leads to the adjoint operator (1.4) as the linearization, so the generalized Hermite polynomial eigenfunctions of \mathbf{B}^* can be key for understanding this intriguing rescaled blow-up dynamics.

2. FUNDAMENTAL SOLUTION AND THE CONVOLUTION

2.1. Fundamental solution and its first properties. In constructing fundamental solutions and corresponding convolutions for (1.1), one can use the fact that formally changing the independent time variable $it \mapsto t$ yields the standard poly-harmonic PDE:

$$(2.1) \quad it \mapsto t \implies u_t = -(-\Delta)^m u \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+.$$

This creates an artificial complex (imaginary) time axis, and nevertheless will help us to restore various spectral properties and other functional details related to (1.1). Of course, unlike the real parabolic case [17], the change in (2.1) implies a well known highly oscillatory properties of the fundamental and other solutions of the Schrödinger equation (1.1), that are not available for its real parabolic counterpart in (2.1).

Thus, by classic PDE theory, given proper initial data $u_0(x)$, the unique solution of the Cauchy problem for (2.1) is given by

$$(2.2) \quad u(t) = b(t) * u_0 \quad \text{for } t > 0,$$

where $b(x, t)$ is the fundamental solution (1.10) of the operator in (2.1). Substituting $b(x, t)$ into (1.1), one obtains the *rescaled kernel* $F(y)$ as a unique solution of a complex

should also be aware of any possible larger context that one’s results could be placed in, as this may lead to the greatest long-term benefit for the result, for the field, and for mathematics as a whole.” [87, p. 633, Tao’s last sentence therein]. [In author’s opinion, Tao’s (ii) statement could definitely serve as a most impressive characterization of a “good mathematics” among dozens of others presented in his paper.] The latter one (ii) somehow helps us to justify including some “risky applications”, which we are not aware how to prove, and even are not sure whether such results can be proved in any remote future; though the authors believe that this stuff should be revealed for the Readers, which might be interested in nowadays or will be later on.

linear ordinary differential equation (ODE) which is the radial restriction of a linear PDE system,

$$(2.3) \quad \mathbf{B}F \equiv -i(-\Delta_y)^m F + \frac{1}{2m} y \cdot \nabla_y F + \frac{N}{2m} F = 0 \quad \text{in } \mathbb{R}^N,$$

so there occurs the linear operator \mathbf{B} given in (1.11). In addition, the kernel F is defined in such a way that, in the sense of distributions (or other suitable L^p -type metrics):

$$(2.4) \quad b(t) * u_0 \rightarrow u_0, \quad t \rightarrow 0^+, \quad \text{or} \quad \int_{\mathbb{R}^N} F(y) \chi(t^{\frac{1}{2m}} y) dy \rightarrow \chi(0) \quad \forall \chi \in C_0^\infty(\mathbb{R}^N),$$

justifying initial data. These define the unique rescaled kernel F .

On the other hand, using the Fourier transform yields the following equivalent representation of F :

$$(2.5) \quad \mathcal{F}(b(\cdot, t))(\omega) = e^{-i|\omega|^{2m}t} \implies \mathcal{F}(F(\cdot))(y) = e^{-i|y|^{2m}}.$$

For $m = 1$, this gives the “Gaussian” exponential profile

$$(2.6) \quad F(y) = \frac{1}{(4\pi i)^{N/2}} e^{\frac{i|y|^2}{4}} \quad (m = 1).$$

It follows that $F(y)$ is highly oscillatory as $y \rightarrow \infty$. In particular, for (2.6), we have

$$(2.7) \quad F(y) = \frac{1}{(4\pi i)^{N/2}} \left[\cos\left(\frac{|y|^2}{4}\right) + i \sin\left(\frac{|y|^2}{4}\right) \right], \quad |F| \equiv \text{const.}, \quad F \notin L^p(\mathbb{R}^N), \quad p \geq 1.$$

For arbitrary $m \geq 2$, the asymptotic behaviour of $F(y)$ for $|y| \gg 1$ is covered by the classic WKB asymptotics. Namely, fixing in (2.3) two main leading terms for the radial kernel $F = F(z)$, for $z = |y| \rightarrow \infty$,

$$(2.8) \quad -i(-1)^m F^{(2m)} + \frac{1}{2m} F' z + \dots = 0$$

yields, in the first approximation, the following exponential asymptotic behaviour:

$$(2.9) \quad F(y) \sim e^{a|y|^\alpha} \implies \alpha = \frac{2m}{2m-1} \quad \text{and} \quad (\alpha a)^{2m-1} = \frac{(-1)^{m+1}i}{2m}.$$

Hence, there exist $2m - 1$ different complex solutions $\{a_k\}$ belonging to a circle in \mathbb{C} :

$$(2.10) \quad |a_k| = z_m = \frac{1}{\alpha} (2m)^{-\frac{1}{2m-1}} < 1 \quad \text{for } m \geq 1.$$

Obviously, we are interested in those roots a_k , for which $\text{Re } a_k \leq 0$. Otherwise these will be exponentially growing oscillatory solutions that will be “too much” non-integrable. It is clear that there exists the purely imaginary root with the main asymptotic oscillatory behaviour at infinity:

$$(2.11) \quad a_0 = z_m i \implies F(y) \sim \cos(z_m |y|^\alpha) + i \sin(z_m |y|^\alpha) \quad \text{as } y \rightarrow \infty.$$

On the other hand, the ODE (2.3) admits solutions with a power decay:

$$(2.12) \quad \frac{1}{2m} F' z + \frac{N}{2m} F + \dots = 0, \quad z = |y| \gg 1 \implies \tilde{F}(y) \sim \frac{C}{|y|^N} \quad \text{as } y \rightarrow \infty.$$

Since $\tilde{F}(y)$ is then are non-oscillatory (for being used as in (2.4)) and, in addition, are “too much” non-integrable as $y \rightarrow \infty$, such asymptotics are not acceptable for the fundamental

kernel $F(y)$. We then arrive at the following simple, but interesting and, in fact, a key property of $F(y)$, which will affect our analysis (especially, in the “nonlinear” cases):

(2.13) all asymptotic components of the rescaled kernel $F(y)$ as $y \rightarrow \infty$ are oscillatory.

2.2. Convolution: a unitary group. Thus, the unique weak solution $u(x, t)$ of the Cauchy problem (1.1) for any data $u_0 \in \mathcal{L}'$ is given by the Poisson-type integral for $t \in \mathbb{R}$:

$$(2.14) \quad u(x, t) = b(t) * u_0 \equiv e^{-i(-\Delta)^m t} u_0 = t^{-\frac{N}{2m}} \int_{\mathbb{R}^N} F((x - z)t^{-\frac{1}{2m}}) u_0(z) dz,$$

where $\{e^{-i(-\Delta)^m t}\}_{t \in \mathbb{R}}$ is the corresponding unitary group. In what follows, for some convenience, we take $t > 0$ only (that implies no trouble in defining the flow in (2.14)), so actually we deal with the semigroup $\{e^{-i(-\Delta)^m t}\}_{t \geq 0}$.

3. DISCRETE REAL SPECTRUM AND EIGENFUNCTIONS OF \mathbf{B}

This section is devoted to some preliminary analysis of spectral properties of the key pair of linear rescaled operators $\{\mathbf{B}, \mathbf{B}^*\}$ that appear after long-time ($t \rightarrow +\infty$) and short-time ($t \rightarrow T^-$) respectively rescaling of the LSE- $2m$ (1.1). In fact, this explains in a reasonably brief manner several necessary key properties of the pair to be used later on. However, we must admit that some of the issues will require a hard work to justify by classic theory. This will take a full separate paper [26]. Nevertheless, we hope that listing a full collection of some involved spectral properties will be convenient for at least some of the Readers, who are interested in general understanding of how this approach works and who do not require full mathematical details.

3.1. First step to the domain of \mathbf{B} in a weighted L^2 -space. We now study spectral properties of the first appeared linear operator \mathbf{B} given in (2.3) in the space $L_\rho^2(\mathbb{R}^N)$ with the exponential weight:

$$(3.1) \quad \rho(y) = e^{-|y|^\alpha} > 0 \quad \text{in } \mathbb{R}^N, \quad \text{where } \alpha = \frac{2m}{2m-1}.$$

By $\langle \cdot, \cdot \rangle$, we denote the standard L^2 -product:

$$(3.2) \quad \langle v, w \rangle = \int_{\mathbb{R}^N} v(y) \overline{w(y)} dy.$$

As customary, $H_\rho^{2m}(\mathbb{R}^N)$ denotes a Hilbert space of functions with the inner product

$$(3.3) \quad \langle v, w \rangle_\rho = \int_{\mathbb{R}^N} \rho(y) \sum_{k=0}^{2m} D^k v(y) \overline{D^k w(y)} dy,$$

where $D^k v$ stands for the vector $\{D^\beta v, |\beta| = k\}$, and the norm

$$(3.4) \quad \|v\|_\rho^2 = \int_{\mathbb{R}^N} \rho(y) \sum_{k=0}^{2m} |D^k v(y)|^2 dy.$$

Obviously, $H_\rho^{2m}(\mathbb{R}^N) \subset L_\rho^2(\mathbb{R}^N) \supset L^2(\mathbb{R}^N)$. Introducing the weighted Sobolev space $H_\rho^{2m}(\mathbb{R}^N)$ is a first step to better understanding the necessary and natural domain of

\mathbf{B} , as stated in the proposition below. However, the space $L_\rho^2(\mathbb{R}^N)$ with the exponentially decaying weight (3.1) is evidently too wide, so we cannot expect any good spectral properties therein. Nevertheless, we now prove the following:

Proposition 3.1. *\mathbf{B} is a bounded linear operator from $H_\rho^{2m}(\mathbb{R}^N)$ to $L_\rho^2(\mathbb{R}^N)$.*

Proof. It follows from (2.3) that $\mathbf{B}v \in L_\rho^2(\mathbb{R}^N)$, if

$$(3.5) \quad \int_{\mathbb{R}^N} \rho(y) |y \cdot \nabla v|^2 dy \leq C \|v\|_\rho^2 \quad \text{for any } v \in H_\rho^{2m}(\mathbb{R}^N),$$

where $C > 0$ is a constant. Let $h \in C^\infty(\mathbb{R}^N)$ be a function such that $h(y) = 1$ for $|y| \geq 2$ and $h(y) = 0$ for $|y| \leq 1$. Since the inequality

$$\int_{\mathbb{R}^N} \rho(y) |y \cdot \nabla [(1 - h(y))v]|^2 dy \leq C_1 \|v\|_\rho^2$$

is obvious, it suffices to show that

$$\int_{\mathbb{R}^N} \rho(y) |y \cdot \nabla (hv)|^2 dy \leq C \|hv\|_\rho^2,$$

i.e., proving (3.5), we can suppose that $v \in C_0^\infty(\mathbb{R}^N)$ vanishes for all $|y| \leq 1$.

Let $(r, \theta_1, \dots, \theta_{N-1})$ be the spherical coordinates in \mathbb{R}^N . Since $|y \cdot \nabla v| \leq r|v_r|$, it suffices to verify that

$$(3.6) \quad \int_0^\infty r^{N+1} |w_1(r)|^2 e^{-r^\alpha} dr \leq C_2 \int_0^\infty r^{N-1} |w_1^{(2m-1)}(r)|^2 e^{-r^\alpha} dr,$$

if the left-hand side in bounded, and apply this estimate with $w_1 = v_r$.

Let $q = N - 1$ or $q = N - 3$, $\gamma = \alpha - 1 = \frac{1}{2m-1}$ and $k \in \{0, 1, \dots, 2m - 2\}$. Then using the inequality

$$\int_0^\infty r^{q+2k\gamma} |w'(r) + r^\gamma w(r)|^2 e^{-r^\alpha} dr \geq 0,$$

integrating by parts again implying that the right-hand side converges, we obtain that

$$\int_0^\infty r^{q+2k\gamma} |w'(r)|^2 e^{-r^\alpha} dr \geq \gamma \int_0^\infty r^{q+2k\gamma+2\gamma} |w(r)|^2 e^{-r^\alpha} dr.$$

Simple iteration implies (3.6) with $C_2 = \gamma^{1-2m}$, completing the proof. \square

The result also follows from a general estimate in [39, Lemma 2.1], which goes back to the Hardy classical inequality [37]. In a similar (or obvious in (ii)) manner, introducing the “adjoint” spaces with the reciprocal weight $L_{\rho^*}^2(\mathbb{R}^N)$, with the weight (3.7).

$$(3.7) \quad \rho^*(y) = \frac{1}{\rho(y)} = e^{|y|^\alpha},$$

we have the following:

Corollary 3.1. *\mathbf{B} is bounded as an operator*

- (i) $\mathbf{B} : H_{\rho^*}^{2m}(\mathbb{R}^N) \rightarrow L_{\rho^*}^2(\mathbb{R}^N)$ and
- (ii) $\mathbf{B} : H_{\rho^*}^{2m}(\mathbb{R}^N) \rightarrow L_\rho^2(\mathbb{R}^N)$.

Remark for $m = 1$. As customary, in the second-order case $m = 1$, there appear some extra possibilities and “symmetries”. Namely, then \mathbf{B} admits a formal symmetric representation

$$(3.8) \quad \mathbf{B} = \mathbf{i} \frac{1}{\kappa} \nabla \cdot (\kappa \nabla) + \frac{N}{2} I, \quad \text{with the “weight” } \kappa(y) = e^{-\mathbf{i} \frac{|y|^2}{4}},$$

in the weighted space L^2_κ with the complex weight $\kappa \neq 0$ and hence with an “indefinite metric”; cf. Azizov–Iokhvidov [4], which we will need to refer to later on a few times at least. We do not know any reasonable application of the complex symmetric form in (3.8). For instance, as usual, the symmetry (3.8) implies the formal orthogonality of eigenfunctions:

$$\langle \psi_\beta, \psi_\gamma \rangle_\kappa \equiv \int \kappa \psi_\beta \psi_\gamma = 0 \quad \text{for } \beta \neq \gamma,$$

but it is not that easy to find suitable applications of this in view of the indefinite metric involved. Anyway, by no means, we are going to rely on this kind of a pseudo-symmetric representation of the operator for $m = 1$, especially, since for $m \geq 2$, this illusive complex symmetry of \mathbf{B} disappears without a trace.

3.2. Group with the infinitesimal generator \mathbf{B} . Before introducing detailed spectral properties of \mathbf{B} , we present a simple derivation of its group for proper weak solutions to be heavily used in what follows.

Thus, the rescaled solution of (1.1) defined as

$$(3.9) \quad w(y, \tau) = t^{\frac{N}{2m}} u\left(y t^{\frac{1}{2m}}, t\right), \quad \text{where } \tau = \ln t \in \mathbb{R} \quad (t > 0),$$

satisfies the necessary rescaled equation

$$(3.10) \quad w_\tau = \mathbf{B} w \quad (\text{the operator } \mathbf{B} \text{ is as in (2.3)}).$$

Then $w(y, \tau)$ solves the CP for (3.10) in $\mathbb{R}^N \times \mathbb{R}_+$ with data at $\tau = 0$ (i.e., at $t = 1$)

$$(3.11) \quad w_0(y) = u(y, 1) \equiv b(y - \cdot, 1) * u_0(\cdot).$$

Rescaling convolution (2.14) yields the following explicit representation of the group with the infinitesimal generator \mathbf{B} :

$$(3.12) \quad w(y, \tau) = e^{\mathbf{B}\tau} u(y, 1) \equiv \int_{\mathbb{R}^N} F\left(y - ze^{-\frac{1}{2m}\tau}\right) u_0(z) dz \quad \text{for } \tau \in \mathbb{R}.$$

Performing another rescaling

$$(3.13) \quad w(y, \tau) = (1+t)^{\frac{N}{2m}} u\left(y(1+t)^{\frac{1}{2m}}, t\right), \quad \text{where } \tau = \ln(1+t) : \mathbb{R}_+ \rightarrow \mathbb{R}_+,$$

we obtain the solution $w(y, \tau)$ of the Cauchy problem for equation (3.10) with initial data $w_0(y) \equiv u_0(y)$. Rescaling (2.14), we deduce a more complicated, but standard (without the relation (3.11)) representation of the semigroup for $\tau \geq 0$

$$(3.14) \quad w(y, \tau) = e^{\mathbf{B}\tau} u_0 \equiv (1 - e^{-\tau})^{-\frac{N}{2m}} \int_{\mathbb{R}^N} F\left((y - ze^{-\frac{1}{2m}\tau})(1 - e^{-\tau})^{-\frac{1}{2m}}\right) u_0(z) dz.$$

By the Hölder inequality (see e.g., (3.24) below), it is easy to see that

$$(3.15) \quad w(\cdot, \tau) \in L^2_\rho(\mathbb{R}^N) \quad \text{for all } \tau > 0 \quad (u_0 \in L^2_{\rho^*}(\mathbb{R}^N)).$$

3.3. Spectral decomposition of \mathbf{B} : a first step to discrete spectrum and eigenfunctions via converging expansion of convolution. We now in a position to make more clear a proper definition of the necessary operator \mathbf{B} to be exploited later on. We then confirm the actual existence and the so-called evolution completeness/closure of some eigenfunctions for initial data $u_0 \in L^2_{\rho^*}(\mathbb{R}^N)$. It is worth to stress out how simple such derivations are here, unlike a more standard spectral and analytic continuation techniques applied in [26] to justify necessary hard properties of resolvent poles and related issues.

It can be derived from the ODE (2.3) for the rescaled kernel $F(|y|)$ and also from (2.5) that the higher-order derivatives of F can be estimated as follows:

$$(3.16) \quad |D^\beta F(y)| \leq C (1 + |y|)^{(\alpha-1)|\beta|} \quad \text{in } \mathbb{R}^N.$$

Actually, according to (2.9), (2.10), a sharper estimate includes the factor $|\alpha z_m|^l$, i.e.,

$$(3.17) \quad |\alpha z_m|^l = (2m)^{-\frac{l}{2m-1}} \rightarrow 0 \quad \text{as } l \rightarrow \infty,$$

so this improves convergence of the series to appear later on.

Consider Taylor's power series of the analytic kernel $F(\cdot)$ on compact subsets $y \in \{|y| \leq L\}$, with an $L \gg 1$,

$$(3.18) \quad F(y - ze^{-\frac{\tau}{2m}}) = \sum_{(\beta)} e^{-\frac{|\beta|\tau}{2m}} \frac{(-1)^{|\beta|}}{\beta!} D^\beta F(y) z^\beta \equiv \sum_{(\beta)} e^{-\frac{|\beta|\tau}{2m}} \frac{1}{\sqrt{\beta!}} \psi_\beta(y) z^\beta,$$

where $z^\beta \equiv z_1^{\beta_1} \dots z_N^{\beta_N}$ and ψ_β are in fact normalized eigenfunctions of \mathbf{B} ; see below. This series converges uniformly on compact subsets in $z \in \mathbb{R}^N$. Indeed, for $|\beta| = l \gg 1$, we have the following approximate estimate of the expansion coefficients:

$$(3.19) \quad \left| \sum_{|\beta|=l} \frac{(-1)^l}{\beta!} D^\beta F(y) z_1^{\beta_1} \dots z_N^{\beta_N} \right| \leq C \frac{1}{l!} (1 + |y|)^{(\alpha-1)l} (1 + |z|)^l,$$

where we have used a rough bound by Stirling's formula (as usual, we often omit the lower-order multipliers $\sim C^l$ in (3.19)):

$$\beta! \geq \left[\left(\frac{l}{N} \right)! \right]^N \sim N^l l!.$$

Finally, we arrive at the following representation of the solution:

$$(3.20) \quad w(y, \tau) = \sum_{(\beta)} e^{-\frac{|\beta|\tau}{2m}} M_\beta(u_0) \psi_\beta(y),$$

where $\lambda_\beta = -\frac{|\beta|}{2m}$ and $\psi_\beta(y)$ are the eigenvalues and eigenfunctions of \mathbf{B} and

$$(3.21) \quad M_\beta(u_0) = \frac{1}{\sqrt{\beta!}} \int_{\mathbb{R}^N} z_1^{\beta_1} \dots z_N^{\beta_N} u_0(z) dz$$

are the corresponding moments of the initial datum w_0 (recall the relation (3.11) between w_0 and u_0). We will show that, in terms of the dual inner product $\langle \cdot, \cdot \rangle$ in $L^2(\mathbb{R}^N)$,

$$(3.22) \quad M_\beta(u_0) = \langle w_0, \psi_\beta^* \rangle,$$

where $\{\psi_\beta^*\}$ are polynomial eigenfunctions of the adjoint operator \mathbf{B}^* to be described in greater detail in Section 4. It is not difficult to check that (3.20) uniformly converges on any compact subset $y \in \{|y| \leq L\}$, since for $l = |\beta| \gg 1$,

$$(3.23) \quad |M_\beta(u_0)\psi_\beta(y)| \leq C \frac{1}{l!} L^{(\alpha-1)l} \int_{\mathbb{R}^N} (1+|z|)^l |u_0(z)| dz,$$

where we estimate the last integral by using the radial variable and the Hölder inequality,

$$(3.24) \quad \begin{aligned} \int_{\mathbb{R}^N} (1+|z|)^l |u_0| dz &= \int_{\mathbb{R}^N} (1+|z|)^l \frac{1}{\sqrt{\rho^*}} \sqrt{\rho^*} |u_0| dz \\ &\leq \sqrt{\int_{\mathbb{R}^N} (1+|z|)^{2l} \frac{1}{\rho^*} dz} \sqrt{\int_{\mathbb{R}^N} \rho^* |u_0|^2 dz}. \end{aligned}$$

The last integral is bounded since $u_0 \in L_{\rho^*}^2$ by the assumptions, while the first integral for very large l can be roughly estimates by Stirling's formula as follows:

$$(3.25) \quad \sqrt{\int_{\mathbb{R}^N} (1+|z|)^{2l} \frac{1}{\rho^*} dz} \sim \sqrt{\int_0^\infty r^{N-1+2l} e^{-r^\alpha} dr} \sim \left(\frac{N+2l}{\alpha e}\right)^{\frac{N+2l}{2\alpha}} \sim \left(\frac{2}{\alpha e}\right)^{\frac{l}{\alpha}} l^{\frac{l}{\alpha}}.$$

Since $\alpha > 1$, the right-hand side is essentially smaller than $l! \sim l^l$ in the denominator in (3.23). Hence, this series converges uniformly and to an analytic solution as expected, and not surprisingly for the LSE.

Recall that (3.14) gave the actual corresponding group $\{T(\tau) = e^{\mathbf{B}\tau}\}_{\tau \in \mathbb{R}}$. Here, we can apply the same expansion analysis as above, which directly determines the eigenfunctions of \mathbf{B}^* . First, by Taylor's expansion in the F -term, we obtain

$$(3.26) \quad w(y, \tau) = \sum_{(\mu)} \frac{(-1)^{|\mu|}}{\mu!} D^\mu F(y(1 - e^{-\tau})^{-\frac{1}{2m}}) (e^\tau - 1)^{-\frac{|\mu|}{2m}} \int_{\mathbb{R}^N} z^\mu w_0(z) dz.$$

Second, applying Taylor expansions for the f -terms in (3.26),

$$(3.27) \quad F(y(1 - e^{-\tau})^{-\frac{1}{2m}}) = \sum_{(\nu)} \frac{1}{\nu!} D^\nu F(0) y^\nu (1 - e^{-\tau})^{-\frac{|\nu|}{2m}}$$

and for functions $(1 - e^{-\tau})^{-\frac{1}{2m}}$ and $(1 - e^{-\tau})^{-\frac{N}{2m}}$ in terms of $e^{-\frac{k\tau}{2m}}$, $k = 0, 1, \dots$, we arrive at a similar representation of the semigroup

$$(3.28) \quad w(y, \tau) = e^{\mathbf{B}\tau} w_0 \equiv \sum_{(\beta)} e^{-\frac{|\beta|\tau}{2m}} \tilde{M}_\beta(w_0) \psi_\beta(y), \quad \tau \geq 0,$$

where the expansion coefficients $\tilde{M}_\beta(w_0)$ are dual products $\langle w_0, \psi_\beta^* \rangle$ with the polynomial eigenfunctions ψ_β^* of the adjoint operator \mathbf{B}^* . Expansion (3.28) determines the adjoint eigenfunctions $\{\psi_\beta^*\}$ that are “orthogonal” to $\{\psi_\beta\}$ in $L^2(\mathbb{R}^N)$ (all products and metrics, with the convention (4.4), to be introduced later on). In Section 4 devoted to the adjoint operator \mathbf{B}^* , we perform a simpler derivation of explicit formulas for polynomials $\{\psi_\beta^*\}$.

Thus, using the expansion (3.28), we are now in a position to present the first proper definition (via its spectral decomposition) of our bounded linear operator in $H_{\rho^*}^{2m}(\mathbb{R}^N)$, which we denote by \mathbb{B} :

$$(3.29) \quad \mathbb{B}w_0 = \frac{d}{d\tau} e^{\mathbf{B}\tau} w_0 \Big|_{\tau=0} \equiv \sum_{(\beta)} \lambda_\beta \tilde{M}_\beta(w_0) \psi_\beta(y) \quad \text{for } w_0 \in H_{\rho^*}^{2m}(\mathbb{R}^N).$$

Actually, \mathbb{B} can be considered as a restriction of a more general operator \mathbf{B} defines in a so-called wider space of closures. Since this extension is not essential for our main applications and \mathbb{B} is sufficient, we postpone this rather technical procedure until Appendix D at the paper end.

Comment on “extended eigenfunctions”. With the definition of our operator

$$(3.30) \quad \mathbb{B} : H_{\rho^*}^{2m}(\mathbb{R}^N) \rightarrow L_{\rho}^2(\mathbb{R}^N)$$

by its expansion (3.29), we face the following technical difficulty. Namely, the eigenfunctions $\{\psi_{\beta}\}$, which actually generate such a \mathbb{B} , *do not belong* to its domain in (3.30), and we refer to them as to *extended eigenfunctions* (i.e., as we will show in Appendix D, belonging to an extended space). However, for simplicity, we continue to call them simply *eigenfunctions*, bearing in mind that, in Appendix D, we are going to construct an extended space of closures, where all ψ_{β} belong to. We then restore the original operator \mathbf{B} rather than its restriction \mathbb{B} , though, as we mentioned, for our main PDE applications, present “spectral theory” of the restriction \mathbb{B} is more than sufficient.

We now summarize the above results concerning the introduced operator \mathbb{B} . Recall that, originally, \mathbb{B} is defined by the rescaled convolution (3.14), with the corresponding space and the domain. For further applications, we restricted \mathbb{B} and defined it via the spectral decomposition such as (3.29), which demanded special topologies for a proper convergence.

Proposition 3.2. *The semigroup expansion series (3.28) for $w_0 \in L_{\rho^*}^2(\mathbb{R}^N)$, and the eigenfunction expansion of \mathbb{B} in (3.29) for $w_0 \in H_{\rho^*}^{2m}(\mathbb{R}^N)$ converge:*

- (i) *uniformly on compact subsets in y , and*
- (ii) *in the mean in $L_{\rho}^2(\mathbb{R}^N)$.*

Proof. (i) has been proved. (ii) The convergence in the mean in $L_{\rho}^2(\mathbb{R}^N)$ is not straightforward for such a bad oscillatory and growing basis functions in Φ . Therefore, estimating the terms in (3.20), we need to include both decaying multipliers in the right-hand side of (3.25), so that the following estimate of a typical integral is key by using Stirling’s asymptotics of the Gamma function:

$$(3.31) \quad \begin{aligned} l^{2l(\frac{1}{\alpha}-1)} \left(\frac{2}{\alpha e}\right)^{\frac{2l}{\alpha}} \int_0^{\infty} z^{N-1} e^{-z^{\alpha}} z^{2(\alpha-1)l} dz &\sim l^{2l(\frac{1}{\alpha}-1)} \left(\frac{2}{\alpha e}\right)^{\frac{2l}{\alpha}} \int_0^{\infty} s^{\frac{N+2(\alpha-1)l}{\alpha}-1} e^{-s} ds \\ &\sim l^{2l(\frac{1}{\alpha}-1)} \left(\frac{2}{\alpha e}\right)^{\frac{2l}{\alpha}} l^{\frac{2(\alpha-1)l}{\alpha}} \left(\frac{2(\alpha-1)}{\alpha e}\right)^{\frac{2(\alpha-1)l}{\alpha}} = (\alpha-1)^{\frac{2(\alpha-1)l}{\alpha}} \left(\frac{2}{\alpha e}\right)^{2l} < \left(\frac{4}{e^2}\right)^l. \end{aligned}$$

Since $\alpha \in (1, 2]$, this exponentially (but not superexponentially as before for the coefficients of (3.20) on compact subsets in y) decaying coefficients guarantee the convergence of the series in $L_{\rho}^2(\mathbb{R}^N)$. The factor in (3.17) is then not necessary for convergence. \square

3.4. Discrete spectrum. Thus, the series (3.29) is a *spectral decomposition* (actually, the eigenfunction expansion for a discrete spectrum) of our linear non self-adjoint operator, which we have denoted by \mathbb{B} . Hence, we treat it as a linear bounded operator given in (3.30), when the eigenfunction expansion (3.28) is defined in two topologies: local uniform

convergence, and strong convergence in the mean in $L^2_\rho(\mathbb{R}^N)$. In this case, the expansion in $L^2_{\rho^*}(\mathbb{R}^N)$, with the same convention on topologies (see (4.4) explaining why $\bar{\psi}_\beta^*$ occur in the last products),

$$(3.32) \quad w \in L^2_{\rho^*}(\mathbb{R}^N) \implies w = \sum c_\beta \psi_\beta, \quad c_\beta = \langle w, \bar{\psi}_\beta^* \rangle,$$

is naturally treated as the eigenfunction series representation of the embedding operator $L^2_{\rho^*}(\mathbb{R}^N) \rightarrow L^2_\rho(\mathbb{R}^N)$ in terms of the eigenfunctions Φ of \mathbb{B} .

For convenience and more systematic understanding the new class of linear operators introduced, we clearly state necessary spectral properties of the operator \mathbb{B} . According to [26], the present operator \mathbb{B} , which was not still properly defined and has been uniquely characterized by its “spectral eigenfunction decomposition” (3.29) only, is not allowed to have the positive part of the point spectrum with polynomial eigenfunctions, as shown in [26], where this is done by a direct introducing “radiation conditions” posed at infinity.

Thus, according to our operator definition (3.29), we ascribe to \mathbb{B} a countable set of numbers, which, for convenience, we continue to call its discrete spectrum:

$$(3.33) \quad \boxed{\sigma(\mathbb{B}) = \left\{ \lambda_\beta = -\frac{|\beta|}{2m}, \quad |\beta| = 0, 1, 2, \dots \right\},}$$

where eigenvalues λ_β have finite multiplicity with eigenfunctions³

$$(3.34) \quad \boxed{\psi_\beta(y) = \frac{(-1)^{|\beta|}}{\sqrt{\beta!}} D^\beta F(y) \equiv \frac{(-1)^{|\beta|}}{\sqrt{\beta!}} \left(\frac{\partial}{\partial y_1} \right)^{\beta_1} \dots \left(\frac{\partial}{\partial y_N} \right)^{\beta_N} F(y).}$$

The existence of such eigenvalues and eigenfunctions is dictated by (3.29). It is worth mentioning that the same follows by applying D^β to the elliptic equation (2.3) (here \mathbf{B} stands for its differential expression, so we are not obliged to use \mathbb{B}): for any β ,

$$(3.35) \quad D^\beta \mathbf{B} F \equiv \mathbf{B} D^\beta F + \frac{|\beta|}{2m} D^\beta F = 0 \implies \left\{ D^\beta F, \lambda_\beta = -\frac{|\beta|}{2m} \right\} \text{ is a pair for } \mathbf{B}.$$

Let us fix some other properties of such extended eigenfunctions:

Lemma 3.1. (i) *The subset of eigenfunctions $\Phi = \{\psi_\beta\}$ is complete in $L^2_\rho(\mathbb{R}^N)$, and*
(ii) *Φ is $L^2_{\rho^*}$ -evolutionary closed in the sense that the eigenfunction expansion (3.28), which converges in the means and uniformly on compact subsets, presents the rescaled solution of the LSE (1.1) for any data $u_0 \in L^2_{\rho^*}(\mathbb{R}^N)$; and*

Remark: towards more general integral evolution equations and rescaled operators \mathbb{B} . As we have mentioned, all the above results can be justified by classic spectral methods associated with the given rescaled differential operators, [26].

However, it is worth mentioning now that all the (i)–(iii) remain valid for more general class of *integral evolution* (pseudo-differential) equations (3.12), where

$$(3.36) \quad \boxed{F(y) \text{ is an arbitrary sufficiently good analytic kernel.}}$$

In other words, (3.12) then do not correspond to any linear PDE. Furthermore, the expansions (3.18) and (3.20) can be also prescribed. Overall, this gives the spectral

³Actually, *extended eigenfunctions*, since $\psi_\beta \notin H^{2m}_{\rho^*}(\mathbb{R}^N)$; a standard meaning *eigenfunctions* of \mathbf{B} to be restored in Appendix D by introducing an *extended space of closures of finite eigenfunction expansions*.

results similar to those in (i)–(iv), where extra efforts to justify the functional topology required are necessary.

In this connection, it is key to emphasize that, in the present most general situation, no extra powerful tools of spectral theory, developed in [26] for the present Schrödinger operators, will be at hand. Then, we will be inevitably attached to a different functional framework, with no visually available operator \mathbb{B} (and the “adjoint” one \mathbb{B}^*).

Returning to Lemma 3.1, note that, as an important characterization of the eigenfunction set Φ , due to (2.13), *all of the eigenfunctions* $\psi_\beta(y)$ satisfy the property (2.13), i.e., these do not have any “non-fast-oscillatory” asymptotic component as $y \rightarrow \infty$ (since (2.12) has been excluded from the kernel $F(y)$). Observe also from (3.34) that, with a proper definition of such *extended (generalized) linear functionals*, to be done in Appendix D, the following can be interpreted as being correct:

$$(3.37) \quad \langle \psi_0, \psi_0^* \rangle_* = \int_{\mathbb{R}^N} \psi_0(y) dy = \int_{\mathbb{R}^N} F(y) dy = 1, \quad \text{since } \psi_0 = F, \psi_0^* = 1.$$

Recall that, in the usual sense, such oscillatory integrals are not properly defined. On the other hand, it also follows from (3.34) that, again in a proper extended linear functional sense (see Appendix D),

$$(3.38) \quad \langle \psi_\beta, \psi_0^* \rangle_* = \int_{\mathbb{R}^N} \psi_\beta(y) dy = 0 \quad \text{for any } |\beta| \geq 1.$$

This is easier to believe in view of the integration by parts, though the oscillatory integrals are not well defined as well. In fact, these equalities express the orthogonality of any ψ_β to the first adjoint eigenfunction $\psi_0^* = 1$ via the dual inner product, defined as extensions of linear functionals prescribed in (3.32); see again Appendix D. The adjoint eigenfunctions are polynomials which form a complete subset in $L_\rho^2(\mathbb{R}^N)$ with the same decaying exponential weight (3.1); see Section 4.

In the second-order case $m = 1$, using the rescaled kernel (2.6) in (3.34) gives the corresponding *generalized Hermite polynomials* $H_\beta(y)$ (given up to normalization constants) via the generating formula:

$$(3.39) \quad \psi_\beta(y) = \frac{1}{(4\pi i)^{N/2}} D^\beta e^{i \frac{|y|^2}{4}} \equiv H_\beta(y) e^{i \frac{|y|^2}{4}}.$$

Note that, in this case, the generalized Hermite polynomials are obtained from the classic ones [9, p. 48] by the change

$$H_\beta(y) = H_\beta^{\text{class.}}(iy) \quad (m = 1).$$

Proof of Lemma 3.1.

(i) Completeness in L_ρ^2 -space. In order to prove completeness in the metric of L_ρ^2 , as in [17, § 2], we suppose that there exists some function G (say, $G \in L^2$), which is orthogonal relative to the inner product in L_ρ^2 to all eigenfunctions, i.e.,

$$\int \rho(y) D^\alpha F(y) G(y) dy = 0 \quad \text{for all } \alpha.$$

Since F is analytic, it implies that

$$\int \rho(y)F(y-x)G(y) dy = 0 \quad \text{for all } x \in \mathbb{R}^N.$$

Consider the Cauchy problem for (1.1) with initial data

$$u_0(x) = \rho(x)G(x) \quad \text{in } \mathbb{R}^N \quad (\text{note that } \rho G \in L^2_{\rho^*}(\mathbb{R}^N)).$$

One can see from the Poisson-type integral (2.14) that the solution exists for all $t \in (0, 1]$. Then $u(x, t)$ is analytic in x . We have

$$u(x, 1) = \int F(x-y)G(y)\rho(y) dy.$$

Therefore, $u(x, 1) \equiv 0$. It follows from the standard uniqueness theorem (see [36] as a guide) that $u(x, 0) = 0$, and $G = 0$.

(ii) Evolution closure in $L^2_{\rho^*}$ has been already proved while studying the convergence of the series (3.20) and (3.28).

This completes the proof of Lemma 3.1. \square

Remark: positive part of the “point spectrum”. Formally, in addition to (3.33), there exists another positive part of the “real spectrum” of the differential form \mathbf{B} (2.3):

$$(3.40) \quad \begin{aligned} \sigma_+(\mathbb{B}) &= \left\{ \lambda_\beta = \frac{N+|\beta|}{2m}, \quad |\beta| = 0, 1, 2, \dots \right\}, \quad \text{where} \\ \psi_\beta^+(y) &= \frac{1}{\sqrt{\beta!}} y^\beta + \dots \quad \text{are } |\beta|\text{-th-degree polynomials.} \end{aligned}$$

We will show how to construct such polynomial eigenfunctions in Section 4, where these are actual eigenfunctions of the adjoint operator \mathbb{B}^* . Though such “pseudo-Hermite” polynomials $\Phi_+ = \{\psi_\beta^+ \in L^2_\rho\}$ actually exist, they are not eigenfunctions since are not available in the given expansions (3.28) as the definition of the operator. In other words, these do not satisfy the necessary *radiation conditions* at infinity, which turn out to be rather non-standard and unusual, [26].

4. Spectrum and polynomial eigenfunctions of the adjoint operator \mathbb{B}^*

4.1. Indefinite metric and domain of the bounded operator \mathbb{B}^* . Using the results obtained above for differential expression \mathbf{B} and its proper restriction \mathbb{B} in (3.29), we now describe in detail the eigenfunctions of a restriction of the “adjoint” operator (1.4), which is still a differential expression and we use the notation \mathbf{B} and not \mathbb{B}^* . \mathbf{B}^* will be obtained via blow-up rescaling (4.6). Note that \mathbf{B}^* is not adjoint in the standard dual metric (3.2) of L^2 , since, as we have seen,

$$(4.1) \quad \hat{\mathbf{B}}^* = (\mathbf{B})_{L^2}^* = -\mathbf{B} + \frac{N}{2m} I \equiv \bar{\mathbf{B}}^*.$$

Curiously, \mathbf{B}^* is then the standard adjoint to \mathbf{B} (as a differential form, in $C_0^\infty(\mathbb{R}^N)$) in the indefinite complex metric (cf. (3.2))

$$(4.2) \quad \langle v, w \rangle_* = \int_{\mathbb{R}^N} v(y)w(y) dy \implies (\mathbf{B})_*^* = \mathbf{B}^*,$$

where the complex conjugation in the second multiplier is not assumed.

In fact, being treated in the metric of L^2 , without a weight as it used to be, such an indefinite metric is not that challenging and even can be partially get rid of for the practical use of eigenfunction expansions; see below. However, some comments are necessary. Firstly, the set of real L^2 -functions

$$E_+ = \{v \in L^2 : \operatorname{Im} v = 0\}$$

is a *positive lineal* (a linear manifold in the field of real numbers)) of the metric, i.e.,

$$\langle v, v \rangle_* > 0 \quad \text{for } v \in E_+ \subset L^2, \quad v \neq 0.$$

The purely imaginary functions,

$$E_- = \{v \in L^2 : \operatorname{Re} v = 0\},$$

define the corresponding *negative lineal*. Therefore, L^2 with this metric is *decomposable*:

$$v = v_+ + v_- \equiv \frac{v(y) + \bar{v}(y)}{2} + \frac{v(y) - \bar{v}(y)}{2}, \quad \text{where } v_{\pm} \in E_{\pm} \implies L^2 = E_+ \oplus E_-.$$

This defines the corresponding positive *majorizing* metric as follows:

$$|\langle v, v \rangle_*| \leq [v, v]_* \equiv \langle v_+, v_+ \rangle_* - \langle v_-, v_- \rangle_*,$$

etc. It should be noted that such a case of the decomposable space with an indefinite metric having a simple majorizing one is treated as rather straightforward; see Azizov–Iokhvidov [4] for linear operators theory in spaces with indefinite metrics⁴.

Though, as we have mentioned, linear operator theory in spaces with indefinite metrics exists for more than half a century, we do not think that the complex indefinite metric in (4.2) creating the necessary pair $\{\mathbf{B}, \mathbf{B}^*\}$ (the operator and its adjoint) can play any role in what follows. On the other hand, as customary, using the metric (4.2) is not that suspicious, since it is necessary *only* for calculating the expansion coefficients according to the standard rule:

$$(4.3) \quad v = \sum c_{\beta} \psi_{\beta} \implies c_{\beta} = \langle v, \psi_{\beta}^* \rangle_*,$$

while all convergence calculus can be continued to be performed in standard metrics. However, to avoid possible future accusations of using non-approved indefinite metrics, we are now back to standard scalar products by noting the following. Since \mathbf{B}^* is shown to have a real point spectrum only, using the standard L^2 -metric instead of (4.2) will only mean replacing the eigenfunctions as follows:

$$(4.4) \quad \psi_{\beta}^* \mapsto \bar{\psi}_{\beta}^*,$$

and we are assuming using this convention any time when necessary and convenient.

⁴Basic results of linear operator theory in spaces with indefinite metrics can be found in Azizov–Iokhvidov’s monograph [4]. It was in 1944, when L.S. Pontryagin published the pioneering paper “Hermitian operators in spaces with indefinite metric” [78]. A new area of operator theory had been formed from Pontryagin’s studies, which, during the time of the WWII, were originated and associated with some missile-type military research (a comment by Yu.S. Ledyayev). This work set by Pontryagin was continued from 1948 and in the 1950s by M.G. Krein [59, 60], I.S. Iokhvidov [40], and others.

In the second-order case $m = 1$, (1.4) has a formal complex symmetric representation

$$(4.5) \quad \mathbf{B}^* = i \frac{1}{\kappa^*} \nabla \cdot (\kappa^* \nabla), \quad \kappa^*(y) = e^{i \frac{|y|^2}{4}},$$

though we do not use this. Similar to \mathbf{B} , we do not know any advantages, which this symmetry in such an indefinite metric can provide. However, for $m \geq 2$, any formal additional symmetry is not available.

For any $m \geq 1$, we again consider \mathbf{B}^* in the weighted space $L_\rho^2(\mathbb{R}^N)$ with the same exponentially decaying weight (3.1), and ascribe to \mathbf{B}^* the domain $H_\rho^{2m}(\mathbb{R}^N)$, which is dense in $L_\rho^2(\mathbb{R}^N)$. As in the previous section, $\mathbf{B}^* : H_\rho^{2m}(\mathbb{R}^N) \rightarrow L_\rho^2(\mathbb{R}^N)$ is shown to be a bounded linear operator.

4.2. Semigroup with infinitesimal generator \mathbf{B}^* . In order to construct the semigroup with the infinitesimal generator \mathbf{B}^* , we use the rescaled variables corresponding to blow-up as $t \rightarrow 1^-$,

$$(4.6) \quad u(x, t) = w(y, \tau), \quad y = \frac{x}{(1-t)^{1/2m}}, \quad \tau = -\ln(1-t) : (0, 1) \rightarrow \mathbb{R}_+.$$

Then w solves the problem

$$(4.7) \quad w_\tau = \mathbf{B}^* w \quad \text{for } \tau > 0, \quad w(0) = u_0 \in L_{\rho^*}^2(\mathbb{R}^N).$$

Rescaling (2.14), we obtain the following explicit representation of the semigroup:

$$(4.8) \quad w(y, \tau) = e^{\mathbf{B}^* \tau} u_0 \equiv (1 - e^{-\tau})^{-\frac{N}{2m}} \int_{\mathbb{R}^N} F\left((ye^{-\frac{1}{2m}\tau} - z)(1 - e^{-\tau})^{-\frac{1}{2m}}\right) u_0(z) dz.$$

4.3. Spectral decomposition and definition of \mathbf{B}^* : using explicit representation of the semigroup. Similar to \mathbf{B} in Section 3, the original rescaled “adjoint” operator \mathbf{B}^* is defined by the convolution (4.8). For the purpose of applications, we will need its restriction defined in terms of its spectral decomposition obtained via the semigroup representation (4.8). Comparing semigroups (4.8) and (3.14), we see that the only difference is in the argument of the rescaled kernel $F(\cdot)$. Therefore, instead of (3.27), we have to use the following expansion:

$$(4.9) \quad F\left((ye^{-\frac{1}{2m}\tau} - z)(1 - e^{-\tau})^{-\frac{1}{2m}}\right) = \sum_{(\gamma)} \frac{D^\gamma F(0)}{\gamma!} (ye^{-\frac{1}{2m}\tau} - z)^\gamma (1 - e^{-\tau})^{-\frac{|\gamma|}{2m}},$$

where $(ye^{-\frac{1}{2m}\tau} - z)^\gamma = \sum_{(0 \leq \delta \leq \gamma)} C_\gamma^\delta e^{-\frac{|\gamma-\delta|}{2m}\tau} y^{\gamma-\delta} (-z)^\delta$.

Then, using both (4.9) in (4.8) yields

$$(4.10) \quad e^{\mathbf{B}^* \tau} u_0 = \sum_{(s \geq 0)} \sum_{(\gamma)} \sum_{(0 \leq \delta \leq \gamma)} e^{-(\frac{|\gamma-\delta|}{2m} + s)\tau} \times (-1)^{|\gamma|} y^{\gamma-\delta} \kappa_s\left(\frac{|\gamma|+N}{2m}\right) \frac{1}{(\gamma-\delta)!} D_\zeta^{\gamma-\delta} \left[\int \left(\frac{1}{\delta!} D^\delta F(\zeta) z^\delta\right) u_0(z) dz \right] \Big|_{\zeta=0}.$$

This is the expansion over the point spectrum of \mathbf{B}^* ,

$$(4.11) \quad w(y, \tau) = e^{\mathbf{B}^* \tau} w_0 = \sum_{(\beta)} e^{-\frac{|\beta|}{2m}\tau} M_\beta^*(u_0) \psi_\beta^*(y),$$

where $\psi_\beta^*(y)$ are finite polynomial eigenfunctions (see their direct derivations below) and the expansion coefficients are

$$M_\beta^*(u_0) = \langle u_0, \psi_\beta \rangle.$$

Convergence of the series (4.11) is studied as in Proposition 3.2. Similar to (3.29), the group representation (4.11) defines the necessary operator \mathbb{B}^* satisfying (3.30) as

$$(4.12) \quad \mathbb{B}^* w_0 = \frac{d}{d\tau} e^{\mathbb{B}^* \tau} w_0 \Big|_{\tau=0} \equiv \sum_{(\beta)} \lambda_\beta \tilde{M}_\beta^*(w_0) \psi_\beta^*(y), \quad w_0 \in H_{\rho^*}^{2m}(\mathbb{R}^N).$$

Similarly to the case of the operator \mathbb{B} at the beginning of the proof of Lemma 3.1, in view of standard regularity properties of linear parabolic flows such as (4.7), the semigroup expansion (4.11) reveals some key auxiliary spectral properties of \mathbb{B}^* :

- (i) the point spectrum is $\sigma(\mathbb{B}^*) = \{\lambda_\beta = -\frac{|\beta|}{2m}\}$, with any λ_β having finite multiplicity;
- (ii) by the definition, there is no continuous spectrum; and
- (iii) polynomial eigenfunctions⁵ $\{\psi_\beta^*(y)\}$ are closed in L_ρ^2 , etc.

In addition, we have to observe that, unlike (2.13) for \mathbb{B} , for the adjoint operator \mathbb{B}^* , the opposite characterization of all the eigenfunctions is in use: all the eigenfunctions

$$(4.13) \quad \psi_\beta^*(y) \text{ of } \mathbb{B}^* \text{ are not oscillatory and are of a “minimal” growth as } y \rightarrow \infty.$$

The last issue of $\psi_\beta^*(y)$ being of a “minimal” growth (since there are many faster other asymptotics that are oscillatory) will be key in the nonlinear setting for the QLSE (1.16). In the linear case $n = 0$, all those notions admit a natural (but still not that easy) standard treatment, so we do not need to stress upon such issues in what follows.

Thus, the definition (4.12) of the operator \mathbb{B} justifies the necessary part of the point spectrum with $\lambda_\beta = -\frac{|\beta|}{2m}$ and excludes its “positive part” (obviously nonexistent in (4.12))

$$(4.14) \quad \lambda_\beta^+ = \frac{N+|\beta|}{2m}, \quad |\beta| \geq 0, \quad \text{with} \quad \psi_\beta^{*+} = \bar{\psi}_\beta.$$

Nevertheless, for convenience, we will continue to refer to (4.13) as a simple, efficient, and actually true way for a correct characterization of necessary eigenfunctions. In other words, the definition of \mathbb{B}^* actually includes special “radiation-like” conditions at infinity, which delete the non-desirable positive spectrum.

4.4. Discrete spectrum and Hermitian polynomial eigenfunctions. Thus, defining \mathbb{B}^* by (4.12), with the discrete spectrum only:

$$(4.15) \quad \sigma(\mathbb{B}^*) = \sigma(\mathbb{B}) = \left\{ \lambda_\beta = -\frac{|\beta|}{2m}, \quad |\beta| = 0, 1, 2, \dots \right\},$$

where all eigenvalues have finite multiplicity, and polynomial extended eigenfunctions to be determined explicitly shortly. Next, as customary, we fix other properties of the adjoint operator in a manner similar to Lemma 3.1.

⁵Again, there are *extended* ones, $\notin H_{\rho^*}^{2m}$, and will restore a usual meaning in Appendix D.

Lemma 4.1. *Under the above hypothesis and conditions:*

- (i) *(Extended) eigenfunctions $\psi_\beta^*(y)$ are polynomials of order $|\beta| \geq 0$;*
- (ii) *The subset of eigenfunctions $\Phi^* = \{\psi_\beta^*\}$ is complete and closed in $L_\rho^2(\mathbb{R}^N)$; and*
- (iii) *Φ^* is $L_{\rho^*}^2$ -evolutionary closed in the sense that the eigenfunction expansion (4.11), which converges in the mean and uniformly on compact subsets, presents the rescaled (according to (4.6)) solution of the LSE (1.1) for any data $u_0 \in L_{\rho^*}^2(\mathbb{R}^N)$; and*

Proof. (i) Construction of polynomial eigenfunctions. Of course, the necessary discrete spectrum (4.15) follows from (4.11). We now intend to show how to obtain these results directly from the differential operator.

Thus, $\psi^*(y)$ is a polynomial. If its degree is k , then

$$\psi^*(y) = \sum_{j=0}^s P_j(y),$$

where $P_j(y)$ is a homogeneous polynomial of degree $k - 2mj$ with $s = \lfloor \frac{k}{2m} \rfloor$, denoting the integer part. Since by the Euler identity

$$-\frac{1}{2m} \sum_{j=1}^N y_j \frac{\partial P_0(y)}{\partial y_j} = -\frac{k}{2m} P_0(y) = \lambda P_0(y),$$

we see that $\lambda = -\frac{k}{2m}$ and $P_0(y)$ may be an arbitrary homogeneous polynomial of degree k . Other polynomials $P_j(y)$ are then defined as follows:

$$P_j(y) = \frac{1}{j!} (\mathbf{i}(-\Delta)^m)^j P_0(y), \quad j = 1, \dots, s.$$

This structure of $\psi^*(y)$ implies the completeness of the set of eigenfunctions in $L_\rho^2(\mathbb{R}^N)$. In the second-order case $m = 1$, this construction leads to the generalized Hermite polynomials, which were already introduced in (3.39). Note that the polynomial structure of adjoint eigenfunctions follows from the expansion (3.28), where the coefficients of initial data $\tilde{M}_\beta(w_0)$ in (3.21) are the dual products of $w_0 \in L_{\rho^*}^2(\mathbb{R}^N)$ and $\tilde{\psi}_\beta^* \in L_\rho^2(\mathbb{R}^N)$. This implies that each $\psi_\beta^*(y)$ is a finite linear combination of elementary polynomials y^γ .

We now fix $P_0(y) = y^\beta$, so that, for (extended) eigenfunctions $\{\psi_\beta\}$ of \mathbb{B} in (3.34), the corresponding adjoint eigenfunctions take the form

$$(4.16) \quad \boxed{\psi_\beta^*(y) = \frac{1}{\sqrt{\beta!}} \left[y^\beta + \sum_{j=1}^{\lfloor \frac{|\beta|}{2m} \rfloor} \frac{1}{j!} (\mathbf{i}(-\Delta)^m)^j y^\beta \right].}$$

We also call (4.16) the *generalized Hermite polynomials*. For $m = 1$, up to normalization constants, these coincide with those given by the classic generating formula (3.39).

(ii) Completeness and closure. This is the well-known fact that polynomials $\{y^\beta\}$, which are higher-order terms in any eigenfunction ψ_β^* , are complete in suitable weighted L^p -spaces; see [51, p. 431]. Closure is associated with the eigenfunction expansion (4.11).

(iii) Evolution closure follows from (4.11). \square

5. Application I: EVOLUTION COMPLETENESS OF Φ IN $L^2_{\rho^*}(\mathbb{R}^N)$, SHARP ESTIMATES IN \mathbb{R}^{N+1}_+ , AND SOME EXTENSIONS

5.1. Linear PDEs. Our first result is about the following refined asymptotic scattering:

Theorem 5.1. *Consider the Cauchy problem (1.1) for $u_0 \in L^2_{\rho^*}(\mathbb{R}^N)$ and $u_0 \neq 0$. Then there exists a finite $l \geq 0$ and a function $\varphi_l(y)$, such that, as $t \rightarrow +\infty$,*

$$(5.1) \quad u(x, t) = t^{-\frac{N+l}{2m}} \left[\varphi_l\left(\frac{x}{t^{1/2m}}\right) + O\left(t^{-\frac{1}{2m}}\right) \right]$$

uniformly on compact sets in $y = \frac{x}{t^{1/2m}}$, where $\varphi_l(y)$ is a nontrivial superposition of extended eigenfunctions $\{\psi_\beta, |\beta| = l\}$ of \mathbb{B} from the corresponding finite-dimensional eigenspace.

Of course, this is a corollary of our convergence analysis of the series (3.20) and (3.28), where l is the minimal multiindex length $|\beta| = l$, for which $\tilde{M}_\beta(w_0) \neq 0$. As in Agmon's classic results for the parabolic case (see zero set applications of advanced Agmon–Ogawa estimates in [11] for parabolic PDEs for $m = 1$), a super-fast decay in (5.1) corresponding to $l = \infty$ implies $u(x, t) \equiv 0$, so that (other topologies are meant)

$$(5.2) \quad |u(x, t)| \leq t^{-K} \text{ as } t \rightarrow \infty \text{ for any } K \gg 1 \implies u = 0.$$

Further extensions of the above classification of $t \rightarrow \infty$ behaviour are rather straightforward for asymptotically small perturbations of the LSE such as

$$(5.3) \quad u_t = -i(-\Delta)^m u + \sum_{(0 \leq |\gamma| < 2m)} a_\gamma(x, t) D^\gamma u \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+,$$

with, say, bounded complex-valued coefficients $\{a_\gamma\}$, which decay sufficiently fast:

$$(5.4) \quad |a_\gamma(x, t)| = o\left(t^{\frac{|\gamma|}{2m}-1}\right) \quad \text{as } t \rightarrow +\infty \text{ for any } 0 \leq |\gamma| \leq 2m-1.$$

We then deal with solutions $u(\cdot, t) \in \tilde{L}^2_\rho(\mathbb{R}^N)$, for which eigenfunction expansions make sense. Then after scaling (3.13), we arrive at the same equation (3.10) with asymptotically (and exponentially if necessary) small perturbations, which can be tackled by the eigenfunction expansion techniques, though some parts of the study can be indeed technical. Some extra efforts are necessary to tackle convergence properties of such series that look rather technical, though can be involved in some places.

5.2. NLSE: on a “centre subspace” behaviour. There are no doubts that, at least partially, the classification results for decaying as $t \rightarrow \infty$ solutions can be extended to the NLSE (1.15). Indeed, the nonlinear term then has the form

$$(5.5) \quad i|u|^{p-1}u \equiv a_0(x, t)u, \quad \text{where } a_0(x, t) = i|u(x, t)|^{p-1} \rightarrow 0 \text{ as } t \rightarrow +\infty$$

on such small solutions. Therefore, one can expect that the perturbation techniques are also effective here.

Agreeing with that and do not performing this routine, but sometimes technical analysis, we would like to stress our attention to the principal fact showing that even this looking purely perturbation approach is not straightforward. Namely, we next formally

show that the NLSE (1.15) can admit some small solutions with a complicated asymptotics corresponding to *centre subspaces* of the rescaled operators.

Bearing in mind the typical linear behaviour (5.1), where $l = 0, 1, 2, \dots$, we perform in (1.15) the standard rescaling as in (3.13),

$$(5.6) \quad u(x, t) = (1+t)^{-\frac{N+l}{2m}} v(y, \tau), \quad y = \frac{x}{(1+t)^{1/2m}}, \quad \tau = \ln(1+t).$$

The rescaled solution $v(y, \tau)$ satisfies the perturbed equation

$$(5.7) \quad v_\tau = \left(\mathbf{B} + \frac{l}{2m} I\right) v + i e^{-\gamma_l \tau} |v|^{p-1} v, \quad \text{where} \quad \gamma_l = \frac{(p-1)(N+l)}{2m} - 1.$$

It follows that there exists a sequence of critical exponents $\{p_l, l = 0, 1, 2, \dots\}$ such that

$$(5.8) \quad \gamma_l = 0 \quad \text{iff} \quad p = p_l = 1 + \frac{2m}{N+l}, \quad l \geq 0.$$

In these critical cases, (5.7) yields the autonomous equation

$$(5.9) \quad v_\tau = \left(\mathbf{B} + \frac{l}{2m} I\right) v + i |v|^{p-1} v.$$

Then, we are looking for solution $v(\cdot, \tau)$ with the behaviour for $\tau \gg 1$ close to the centre subspace of the linearized operator $\mathbf{B} + \frac{l}{2m} I$. Such a centre subspace asymptotic dominance assumes that in the eigenfunction expansion of the solution

$$(5.10) \quad v(y, \tau) = \sum_{(\beta)} c_\beta(\tau) \psi_\beta(y)$$

the leading term for $\tau \gg 1$ corresponds to an eigenfunction φ_l of \mathbf{B} ,

$$\left(\mathbf{B} + \frac{l}{2m} I\right) \varphi_l = 0,$$

i.e., φ_l belongs to the centre subspace of the linearized operator $\mathbf{B} + \frac{l}{2m} I$ in the perturbed equation (5.9). Hence, we suppose that

$$(5.11) \quad v(\tau) = a_l(\tau) [\varphi_l + o(1)] \quad \text{as} \quad \tau \rightarrow +\infty$$

and this asymptotic equality can be differentiated in y and τ . Then the equation for the leading expansion coefficient $a_l(\tau)$ takes the form (recall the convention (4.4) for the metric)

$$(5.12) \quad \dot{a}_l = i |a_l|^{p_l-1} a_l [c_l + o(1)] \quad \text{for} \quad \tau \gg 1, \quad \text{where} \quad c_l = \langle |\varphi_l|^{p_l-1} \varphi_l, \varphi_l^* \rangle,$$

and $c_l \neq 0$ as should be assumed. Here, for the first (and the last) time, we need the values of generalized *extended linear functional* to be introduced in Appendix D. We must admit that it is not easy to evaluate such coefficients c_l , even numerically in 1D. Moreover, we still do not know any efficient method to get these values, so we will need further speculations.

However, to confirm that such a formal analysis actually makes sense, we present a simple *explicit* example of such solutions on a centre subspace (in fact, on a manifold).

Example: explicit centre subspace periodic solutions for $m = 1, l = 0$. Obviously, the simpler case is $l = 0$ and $m = 1$, where the eigenspace is 1D, so that $\varphi_0 = F$ and hence, by (2.6),

$$(5.13) \quad |F(y)| = b_1 = \frac{1}{(4\pi)^{N/2}} \implies c_0 = \langle |F|^{p_0-1} F, \psi_0^* \rangle = b_1^{\frac{2m}{N}},$$

so that c_0 is real. We have used the convention $\int F = 1$, which, for $N = 1$, holds in the usual sense, since the improper integral converges. Then substituting into (5.9), with $l = 0$, yields the following explicit solution:

$$(5.14) \quad v(y, \tau) = a_0(\tau)F(y) \implies \dot{a}_0 = i |a_0|^{\frac{2}{N}} a_0 \implies \frac{d}{d\tau} |a_0(\tau)|^2 = 0,$$

i.e., this explicit centre subspace behaviour is a *periodic orbit*. Note that this is not an $L^2(\mathbb{R}^N)$ or any $L^p(\mathbb{R}^N)$ solution, since $F(y) \equiv \psi(y)$ is the first (extended) eigenfunction of the operator \mathbb{B} , and also a standard eigenfunction of \mathbf{B} in the extended space of eigenfunction expansion closures; see Appendix D. Therefore, we do not know whether a centre subspace solution (5.14) can have a stable orbit connection with more customary L^2 and other solutions of Schrödinger equation theory.

For general $m \geq 2$ and arbitrary $l \geq 0$, we need further arguments concerning the coefficients c_l in (5.12). For instance, using the continuity argument with respect to p , we again observe “almost” real values, since:

$$(5.15) \quad c_l = 1 \text{ by orthonormality, if } p_l = 1, \text{ so } c_l \approx 1 \text{ if } p_l \approx 1 \text{ for all } l \gg 1.$$

However, such an asymptotic result assumes a technical proof, which falls out of the scope of the present analysis. Overall, for real coefficients c_l , the system (5.12) is Hamiltonian,

$$\frac{d}{d\tau} |a_l(\tau)|^2 = 0,$$

so, as above for $l = 0$, it describes a *periodic behaviour* close to the centre subspace.

For complex valued c_l , such a centre subspace behaviour can be more complicated. For instance, for $c_l = i \hat{c}_l$, with $\hat{c}_l > 0$ (a rather hypothetical situation to be used as an illustration only), integrating (5.12) yields the following decaying functions:

$$(5.16) \quad a_l(\tau) = C_l \tau^{-\frac{1}{p_l-1}} (1 + o(1)) \quad \text{as } \tau \rightarrow +\infty,$$

where $C_l \in \mathbb{C}$ is a constant. In terms of the original (x, t, u) -variables, such a behaviour takes a form of a logarithmically perturbed linearized pattern

$$(5.17) \quad u(x, t) = C_l (t \ln t)^{-\frac{N+l}{2m}} \left[\varphi_l\left(\frac{x}{t^{1/2m}}\right) + o(1) \right] \quad \text{as } t \rightarrow +\infty.$$

As we have mentioned, similar asymptotic patterns can be constructed for the “stable”, *defocusing NLSE*

$$(5.18) \quad u_t = -i(-\Delta)^m u - i|u|^{p-1}u \quad \text{in } \mathbb{R}^N \times \mathbb{R} \quad (p > 1);$$

see [48, 88, 92, 95] for key references and results concerning (5.18) for $m = 1$, as well as recent papers [5, 6, 33, 68, 72, 73, 101] (and references/short surveys therein) for $m = 2$, i.e., for the *biharmonic* nonlinear Schrödinger equation

$$(5.19) \quad i u_t + \Delta^2 u = \pm |u|^{p-1}u \quad \text{in } \mathbb{R}^N \times \mathbb{R}.$$

For (5.18), such a centre subspace approach looks even more promising than for the unstable PDE (1.15) admitting also blow-up in these ranges. More flexibility is added when replacing the nonlinear term by a more general one,

$$\pm i |u|^{p-1}u \mapsto d |u|^{p-1}u, \quad \text{where } d = a + ib \in \mathbb{C}, \quad ab \neq 0.$$

A rigorous justification of the centre manifold-like patterns of a periodic or (5.17)-type is a difficult open problem, which we do not touch here. Notice that even existence of an invariant manifold (in which functional setting?—in an extended space of closures as in Appendix D?) is a very difficult problem for such Hermitian spectral theory dealing with the pair $\{\mathbb{B}, \mathbb{B}^*\}$. This is regardless the good spectral properties of \mathbb{B} listed in Lemma 3.1 and also its sectorial setting in the topology of l_ρ^2 in Proposition D.1, which however suggest a certain confidence that this behaviour can be verified by using the powerful machinery of classic invariant manifold theory, [62].

6. Applications II and III: LOCAL STRUCTURE OF NODAL SETS AND UNIQUE CONTINUATION

6.1. Application II: blow-up formation of multiple zeros for linear PDEs (a Sturmian theory). Next, we arrive at the following classification of zeros of solutions of the LSE (1.1):

Theorem 6.1. *Consider the Cauchy problem (1.1) for $u_0 \in L_{\rho^*}^2(\mathbb{R}^N)$ and $u_0 \neq 0$. Assume that the corresponding solution $u(x, t)$ creates a zero at a point $(0, T)$, i.e., $u(0, T) = 0$. Then there exists a finite $l \geq 1$ and a generalized Hermite polynomial $\varphi_\beta^*(y)$ such that*

$$(6.1) \quad u(x, t) = (T - t)^{\frac{l}{2m}} \left[\varphi_l^* \left(\frac{x}{(T-t)^{1/2m}} \right) + o(1) \right] \quad \text{as } t \rightarrow T^-$$

uniformly on compact sets in $y = \frac{x}{(T-t)^{1/2m}}$, where $\varphi_l^(y) \not\equiv 0$ is a superposition of polynomial (extended) eigenfunctions $\{\psi_\beta^*, |\beta| = l\}$ of \mathbb{B}^* from the corresponding eigenspace $\ker(\mathbb{B}^* - \frac{l}{2m})$.*

Since $\varphi_l(y)$ is a generalized Hermite polynomial, the multiple zero of $\operatorname{Re} u(x, t)$ (or, equivalently, $\operatorname{Im} u(x, t)$) occurs at the point $(0, T^-)$ by “blow-up focusing” of several zero-surfaces $\{x_\gamma(t)\}$ of $\operatorname{Re} \varphi_l^* \left(\frac{x}{(T-t)^{1/2m}} \right)$, which move according to the scaling blow-up law

$$(6.2) \quad x_\gamma(t) \sim (T - t)^{\frac{1}{2m}} \rightarrow 0 \quad \text{as } t \rightarrow T^- \quad (|\gamma| \leq l).$$

The result (6.1) follows from the series (4.11), for which the rescaling (4.6) is performed relative to the time moment T rather than 1. In a natural sense, the countable family of the types of asymptotics (6.1) describes the sharp “micro-turbulent” structure of the PDE (1.1), since, by evolution completeness, on smaller space-time scales, the solution behaviour is trivial (a constant one mostly). In other words, (6.1) exhausts all possible micro configurations that can be created by the LSE (and also by many other related semi-linear and quasilinear PDEs admitting similar blow-up rescaling and Hermitian spectral properties; see below). See also [24] for parabolic and other real-valued PDEs.

6.2. Application III: Unique continuation. Various classic and other well-known new unique continuation results for linear and nonlinear Schrödinger-type PDEs can be found in [16, 41, 42, 49], where further references are available. These directions on uniqueness PDE theory have their origins in many principal works in the twentieth century including such a classic path as Holmgren (the starting point, 1901)–Carleman (1933)–Myshkis

(1948)–Plis (1954)–Calderon (1958)–Agmon–Nirenberg... Here we present an example of a slightly different type of a “blow-up micro-scale uniqueness” study based on the spectral properties of \mathbb{B}^* , which is responsible for blow-up scaling of the PDE.

Thus, all the types of nodal sets of zeros for the LSE (1.1) are exhausted by the zero structures of Re or Im of all the generalized Hermite polynomials $\Phi^* = \{\psi_\beta^*(y)\}$ given by (4.16) including arbitrary linear combinations on all the eigenspaces. We fix this in the following rather unusual unique continuation theorem:

Corollary 6.1. *Let under the hypotheses of Theorem 6.1, the nodal set of the real part (or, equally, of the imaginary part) of the solution*

$$(6.3) \quad \mathcal{N}(u) = \{x \in \mathbb{R}^N, t \in \mathbb{R} : \operatorname{Re} u(x, t) = 0\}$$

has a nontrivial component that evolves as $t \rightarrow T^-$ in a manner that does not asymptotically match the zero sets of any finite linear combinations of the real parts of the generalized Hermite polynomials from Φ^ . Then*

$$(6.4) \quad u(x, t) \equiv 0 \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}.$$

Of course, this is just a conventional version of the uniqueness result that is based on the eigenfunction expansion (4.11), which can be expressed in a different and more reliable manner. For instance, e.g., depending on l and also much on $m \geq 2$, for the case, where the resulting polynomial $\operatorname{Re} \varphi_l^*(y)$ does not change sign at all (so that (6.3) is locally empty), we then have to postulate just the asymptotic behaviour such as (6.1). Then the alternative (and more correct and universal) sounding of the unique continuation result will be as follows:

$$(6.5) \quad \text{if } u(x, t) \text{ violates any of non-trivial asymptotics (6.1) near zero, then } u \equiv 0.$$

Recall that, typically, for the real-valued evolution linear or nonlinear PDEs with interior regularity, unique continuation theorems stated in the pointwise sense deals with zeros of infinite order in the following manner: if, in a natural integral mean sense,

$$(6.6) \quad u \text{ has an infinite-order zero at } (0, T), \text{ then } u(x, t) \equiv 0;$$

see [11] and [24] for further references and results for parabolic PDEs (such results are also known for the LSEs and are proved by iterating Carleman’s classic estimates). Of course, (6.6) becomes trivial for analytic solutions (though extensions to smooth non-analytic ones along the lines discussed below makes deep sense), so that we present a new pointwise uniqueness version (6.3), which looks not that trivial.

It is natural to expect that the above classification of all the possible zeros remains for the perturbed LSEs such as (5.3), with arbitrary bounded coefficients $\{a_\gamma\}$. We then need to assume that $u(x, t)$ is locally good enough close to the point $(0, T^-)$, and at least, $u(\cdot, t) \in \tilde{L}_\rho^2(\mathbb{R}^N)$ (see details in Appendix D), so we can use the corresponding eigenfunction expansions endowed with a strong enough topology of convergence on compact subsets, which are sufficient to detect and identify the zero structure of solutions. At least, we need convergence a.e., which is guaranteed by the L_ρ^2 -metric. Nevertheless, the pointwise sense of such expansions will possibly demand some extra hypotheses that are

not discussed here. It is known that, even in the parabolic case, such a Sturmian theory on zero sets leads to a number of technical difficulties; see [24], where further references are given and applications to other classes of PDEs are discussed. Note also that such extensions can be applied to related partial differential inequalities (PDIs), e.g., for

$$(6.7) \quad |u_t + i(-\Delta)^m u| \leq C(|u| + |\nabla u| + \dots + |D^{2m-1}u|),$$

where $C > 0$ is a constant. One can see that the right-hand side is always negligible after rescaling (4.6), so it does not affect the asymptotic zero classification (6.1).

6.3. The NLSE: similar local zero set behaviour. For the NLSE (1.15), the local zero evolution remains unchanged since no centre subspace patterns are available. Namely, assuming again that a zero occurs at $(0, T)$, we perform for (1.15) the scaling

$$(6.8) \quad u(x, t) = v(y, \tau), \quad y = \frac{x}{(T-t)^{1/2m}}, \quad \tau = -\ln(T-t) \implies v_\tau = \mathbf{B}^* v \pm e^{-\tau} i |v|^{p-1} v.$$

In other words, the nonlinear term near the zero always creates an exponentially small as $\tau \rightarrow +\infty$ perturbation of the dynamical system for the expansion coefficients. Hence, it is very unlikely that this can somehow essentially affect the local zero structure near $(0, T)$. Recall again that a rigorous analysis is rather involved even in simpler parabolic cases, [17, 24]. Of course, close to the zeros, the NLSE (1.15) falls into the scope of the PDI (6.7).

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APPENDIX A. “SCHRÖDINGER EQUATIONS” ARE MOST POPULAR IN THE TWENTY FIRST CENTURY ACCORDING TO MathSciNet

We have claimed in Section 1 that Schrödinger PDEs are most popular currently in mathematical PDE theory. To “prove” this statement, we present below Table 1 on citations of the different types of equations in the MathSciNet. One can get these results by typing in the box Title the corresponding type of equations, i.e., “hyperbolic (elliptic, parabolic, etc.) equation”. Each resulting page in MathSciNet contains a variable number of papers (in average, it is about 20, but can be more than that), so for convenience we count the number of pages. Naturally, at this moment, we put a blind eye on the obvious fact that there can be, and indeed many, papers on various PDEs, whose title do not contain those two words. Anyway, we believe that this is a reliable statistics and the table reflects some true information.

Thus, the absolute winners for the total number of papers (via “All matches”), which have in the title “... equation” are “elliptic” and “parabolic” ones (recall that the MathSciNet operated that time with about 3.5 million papers, written by about 200 000 mathematicians). However, during two years 2007–08, more papers were published for “Schrödinger” ones: listing all of them takes about the record **25!** pages. This adds extra bits to our motivation of developing a refined spectral theory for the Schrödinger PDE (1.1).

TABLE 1. Citations in MathSciNet, 8th September, 2008

PDE type in the title	All matches	In 2007–08
“hyperbolic equation”	3321	~ 5 pages
“elliptic equation”	7118	~ 19 pages
“parabolic equation”	6972	~ 18 pages
“Schrödinger equation”	5264	~ 25! pages

APPENDIX B. Application IV: TOWARDS BOUNDARY POINT REGULARITY THEORY

We now present another application of a refined $\{\mathbb{B}, \mathbb{B}^*\}$ -spectral theory to the problem of regularity of *boundary characteristic points* for Schrödinger equations such as (1.2) and (1.1). These regularity issues were always in the core of general potential theory, which in its turn represents one of the most classic directions of linear and nonlinear PDE theory initiated already by Dirichlet himself in 1820s. We refer to Maz’ya’s monographs with collaborators [56, 57] for the history and key results on elliptic PDEs, as well as to recent surveys in [25, 27] devoted more to parabolic PDEs, whose approaches and results will be essentially used later on.

Of course, there are many very strong and classic boundary regularity results for Schrödinger equations (1.2) and (1.1), which are explained in several key papers mentioned in the introduction. In particular, Schrödinger (or similar and often equivalent beam-type) equations in *non-cylindrical domains* have been studied in the 1960s by J.-L. Lions and E. Magenes, and by G.A. Pozzi, and later on by T. Gazenave and others. We refer to papers [3, 8, 12], where further references and results can be found. The existence-regularity results therein in principal cannot treat *characteristic* boundary points, where, as we show, the continuity of the solutions is rather tricky. We hope that our brief regularity exposition based on spectral theory of the pair $\{\mathbb{B}, \mathbb{B}^*\}$ will insert some new features to this classic area, which were not observed before.

B.1. Regular boundary points and general asymptotic problem. Without loss of generality, in order to explain the main ingredients of the boundary point regularity, we consider the simplest case $m = N = 1$, i.e., the 1D *second-order Schrödinger equation*:

$$(B.1) \quad u_t = i u_{xx} \quad \text{in } Q_0,$$

where Q_0 is a typical domain, for which $(0, 0)$ is a *characteristic point*, i.e., the straight line $\{t = 0\}$ touches the lateral boundary of Q_0 . We define the *backward parabola*⁶ Q_0 as follows, assuming that it has a single finite right-hand lateral boundary (we will see why this is necessary):

$$(B.2) \quad Q_0 = \{-\infty < x < R(t), \quad t \in (-1, 0)\},$$

where $R(t) > 0$ for all $t \in [-1, 0)$ is a given sufficiently smooth function satisfying

$$(B.3) \quad R(t) \rightarrow 0^+ \quad \text{as } t \rightarrow 0^-.$$

Finally, we pose the Dirichlet boundary condition:

$$(B.4) \quad u(R(t), t) = 0 \quad \text{for } t \in (-1, 0),$$

⁶In \mathbb{R}^N , it is a *backward paraboloid* with a quite similar study as for $N = 1$, though, of course, it becomes more involved; see those typical boundary regularity features in [28] to 3D Navier–Stokes equations.

and prescribe bounded smooth L^2 -initial data at $t = -1$ (without loss of generality, we allow $u_0(x)$ too decay exponentially fast as $x \rightarrow -\infty$ or to even to be compactly supported):

$$(B.5) \quad u(x, -1) = u_0(x) \quad \text{for } x \in (-\infty, R(-1)).$$

Overall, (B.1)–(B.4) is a well-posed initial-boundary value problem, and we assume that it possesses a classic bounded solution up to the characteristic moment $t = 0^-$.

Thus, the point $(0, 0)$ is called *regular* in Wiener's classic sense if

$$(B.6) \quad u(0, 0) = 0$$

for any such data u_0 , i.e., there is continuity along the boundary, and *irregular* otherwise (if (B.6) fails for some u_0). Our main goal is to show how to answer the following question:

$$(B.7) \quad \boxed{\text{for which lateral boundaries given by } R(t), (0, 0) \text{ is regular (irregular).}}$$

In fact, this follows the canonical regularity statement by I.G. Petrovskii in 1934–35 [75, 76], who almost completely solved the boundary regularity problem for the *heat equation*

$$(B.8) \quad u_t = u_{xx}.$$

This led to his famous “log log backward parabola”, meaning, in particular, the following remarkable and delicate results:

$$(B.9) \quad \boxed{\begin{array}{ll} \text{(i) } R(t) = 2\sqrt{-t} \sqrt{\ln |\ln(-t)|} \implies (0, 0) \text{ is regular, and} \\ \text{(ii) } R(t) = 2(1 + \varepsilon)\sqrt{-t} \sqrt{\ln |\ln(-t)|}, \varepsilon > 0 \implies (0, 0) \text{ is irregular.} \end{array}}$$

In what follows, unlike many strong well-known results and approaches, we follow an asymptotic *matching blow-up* approach to the present regularity problem, which was developed in [25] and [27] for higher-order linear and nonlinear parabolic PDEs, respectively.

Actually, of course, we are solving a more general problem on the asymptotic behaviour of solutions $u(x, t)$ as $x \rightarrow 0$ and $t \rightarrow 0^-$, so our goal is as follows:

$$(B.10) \quad \boxed{\text{to describe sharp asymptotics of } u(x, t) \text{ at the blow-up point } (0, 0^-),}$$

and essentially detect their dependence on the function $R(t)$ defining the right-hand lateral boundary. As usual, once (B.10) has been solved, one can check the regularity property (B.7).

B.2. A comment: two boundary conditions must be non-Hamiltonian. Let us comment on the non-symmetric shape of the domain in (B.2). Using the symmetric shape (as can be done for the heat equation (B.8) or for the bi-harmonic one (1.9))

$$(B.11) \quad \hat{Q}_0 = \{-R(t) < x < R(t), \quad t \in (-1, 0)\}$$

is not possible. Indeed, since (B.1) with the Dirichlet conditions on the lateral boundary of \hat{Q}_0 is a Hamiltonian system with the L^2 -conservation, as $t \rightarrow 0^-$, one observes the concentration of this L^2 -energy onto shrinking-to-zero x -intervals, so that, obviously,

$$(B.12) \quad \text{for domains (B.11) and conservative boundary conditions, } (0, 0) \text{ is irregular,}$$

that eliminated the regularity issue at all. On the other hand, the symmetric domains such as (B.11) are admitted if the boundary conditions on the lateral boundary violate the Hamiltonian

(symplectic) L^2 -conservation property. It is not that easy to find such conditions for the second-order equation (B.1). For instance, these could be the *Robin ones* (a third kind condition) at $x = -R(t)$,

$$u_x + \sigma u = 0 \quad \text{for some } \sigma \in \mathbb{C},$$

and then the regularity would mean the continuity at $(0,0)$ along the right-hand boundary.

For the fourth-order LSE,

$$(B.13) \quad u_t = -i u_{xxxx},$$

a domain (B.11) also requires non-Hamiltonian lateral boundary conditions. By the identity

$$(B.14) \quad \frac{d}{dt} \int |u|^2 = i \left([\bar{u}_{xxx} u - u_{xxx} \bar{u}]_{-R(t)}^{R(t)} + [u_{xx} \bar{u}_x - \bar{u}_{xx} u_x]_{-R(t)}^{R(t)} \right),$$

one can see that the homogeneous Dirichlet conditions

$$u = u_x = 0 \quad \text{at } x = \pm R(t), \quad t \in (-1, 0),$$

are indeed Hamiltonian, the L^2 -norm of $u(\cdot, t)$ is preserved in such domains shrinking to a point, so $(0,0)$ is always irregular. The same is true for the Navier-type conditions

$$u = u_{xx} = 0 \quad \text{at } x = \pm R(t).$$

Again, by (B.14), there exists the L^2 -conservation, so the vertex regularity problem makes no sense and $(0,0)$ is irregular for any nontrivial initial data.

On the other hand, the following conditions

$$(B.15) \quad u = u_{xxx} = 0 \quad \text{at } x = \pm R(t),$$

in general, violate the L^2 -conservation on shrinking domains as $t \rightarrow 0^-$, so the characteristic boundary regularity problem makes sense. We can apply our “blow-up” scaling-matching approach to study regularity of the vertex for conditions like in (B.15) or others of higher-order Robin-kind (but this will require a different boundary layer theory, see below).

The boundary point regularity approach proposed here also covers the problem (B.13), (B.15) and other $2m$ th-order ones with various (non-symplectic) boundary data.

B.3. Introducing slow growing factor $\varphi(\tau)$. Thus, we return to the canonical (and indeed looking very simple) LSE-2 (B.1) in the one-sided domain (B.2), with all conditions already specified. Then, similar to (B.9), we introduce a one-sided backward parabola at $(0,0)$ given by the function

$$(B.16) \quad R(t) = \sqrt{-t} \varphi(\tau), \quad \text{where } \tau = -\ln(-t) \rightarrow +\infty \quad \text{as } t \rightarrow 0^-.$$

In Petrovskii’s criterion (B.9),

$$\varphi(\tau) \sim 2\sqrt{\ln \tau} \quad \text{as } \tau \rightarrow +\infty,$$

so that $\varphi(\tau)$ is an unknown slow growing function satisfying

$$(B.17) \quad \varphi(\tau) \rightarrow +\infty, \quad \varphi'(\tau) \rightarrow 0, \quad \text{and} \quad \frac{\varphi'(\tau)}{\varphi(\tau)} \rightarrow 0 \quad \text{as } \tau \rightarrow +\infty.$$

Moreover, as a sharper characterization of the above class of *slow growing functions*, we use the following criterion:

$$(B.18) \quad \left(\frac{\varphi(\tau)}{\varphi'(\tau)} \right)' \rightarrow \infty \quad \text{as } \tau \rightarrow +\infty \quad (\varphi'(\tau) \neq 0).$$

This is a typical condition in blow-up analysis distinguishing classes of exponential (the limit is 0), power-like (a constant $\neq 0$), and slow-growing functions. See [83, pp. 390-400], where in

Lemma 1 on p. 400, extra properties of slow-growing functions (B.18) are proved to be used later on.

B.4. First kernel scaling. By (B.16), we perform the similarity scaling

$$(B.19) \quad u(x, t) = v(y, \tau), \quad \text{where} \quad y = \frac{x}{\sqrt{-t}}.$$

Then the rescaled function $v(y, \tau)$ solves the rescaled equation

$$(B.20) \quad \begin{cases} v_\tau = \mathbf{B}^* v \equiv i v_{yy} - \frac{1}{2} y v_y & \text{in } Q_0 = \{-\infty < y < \varphi(\tau), \tau > 0\}, \\ v = 0 & \text{at } y = \varphi(\tau), \tau \geq 0, \\ v(0, y) = v_0(y) \equiv u_0(y) & \text{on } (-\infty, R(-1)), \end{cases}$$

where, by obvious (blow-up micro-scale) reasons, the rescaled differential expression \mathbf{B}^* , defined as in (1.4) for $m = N = 1$, appears. In view of the assumed divergence (B.17), it follows that our final analysis will essentially depend on the spectral properties of the corresponding restricted linear operator \mathbb{B}^* on the whole line $y \in \mathbb{R}$, i.e., we arrive at the necessity of Hermitian spectral theory developed above. Note that, in particular, this differs our regularity analysis from several well-known ones such as Kondrat'ev's classic results of the 1966–67 [52, 53] (see a later survey [54]), where, as a rule, the rescaled boundary remains asymptotically fixed, which is key for using the spectral properties of the bundles of linear operators in locally compact domains. We will present further more detailed comments on that below.

B.5. Regularity of a fixed backward parabolae is not obvious. First of all, we need to comment on the regularity of the vertex $(0, 0)$ of the *backward fundamental parabolae*:

$$(B.21) \quad R(t) = l\sqrt{-t}, \quad \text{i.e.,} \quad \varphi(\tau) \equiv l = \text{const.} > 0.$$

Then the problem (B.20) is considered on the fixed unbounded interval

$$(B.22) \quad I_l = \{-\infty < y < l\},$$

so that the final conclusion entirely depends on spectral properties of \mathbf{B}^* in I_l with Dirichlet boundary conditions. Since we need a sharp bound on the first eigenvalue, the clear conclusion on regularity/irregularity becomes rather involved, where numerics are necessary to fix final details. In addition, as we pointed out, in more general setting for the fundamental backward paraboloids in \mathbb{R}^N , the existence, uniqueness, and regularity of solutions in Sobolev spaces was proved in a number of papers such as [69, 70, 71, 21], etc. Note that in [70, p. 45], the zero boundary data were understood in the *mean sense* (i.e., in the L^2 -sense along a sequence of smooth internal contours “converging” to the boundary).

Note that it is not obvious at all that the spectrum of \mathbf{B}^* on I_l (with a standard L^2_ρ -setting as $y \rightarrow -\infty$) is real. However, one can expect that, by a continuity argument, the “first” eigenvalue $\lambda_0 = \lambda_0(l)$ depending on l satisfies

$$(B.23) \quad \lambda_0(l) \rightarrow 0 \quad \text{as} \quad l \rightarrow +\infty.$$

This reminds a standard asymptotic fact from classic perturbation theory of linear operators (see Kato [46]) that the spectrum of \mathbf{B}^* in I_l approaches as $l \rightarrow +\infty$ that in $L^2_\rho(\mathbb{R})$, according to Lemma 4.1. Then along with (B.23), one can also expect convergence of the first eigenfunction:

$$(B.24) \quad \psi^*(y; l) \rightarrow \psi_0^*(y) \equiv 1 \quad \text{as} \quad l \rightarrow +\infty.$$

Notice that (B.24) contains some features of a *boundary layer* that occurs as $l \rightarrow +\infty$, which we will be use in the non-stationary limit $\varphi(\tau) \rightarrow +\infty$ as $\tau \rightarrow +\infty$.

Overall, the limit (B.23) reflects the possibility for $(0, 0)$ to be regular or irregular for different values of $l > 0$ depending of the sign of $\operatorname{Re} \lambda_0(l)$ (or $\lambda_0(l)$ itself, provided that it is real). In other words, we conclude as follows: if the limit (B.23) is oscillatory, then the backward parabolae (B.21) can be regular or irregular. Note that this happens for the bi-harmonic equation (1.9), where the corresponding fixed parabolae with

$$R(t) = (-t)^{\frac{1}{4}}l$$

is regular for $l = 4$ but is irregular for $l = 5$. These conclusions for (1.9) were fully justified numerically only, [25, § 6].

Thus, the regularity analysis of the backward parabolae (B.21) for the operator (B.1) remains open, and its complete solution cannot be done without using enhanced numerical methods. Nevertheless, despite such a theoretical gap for constant l 's, we proceed to study the regularity for unbounded functions (B.17), which promises even greater mathematical challenge.

B.6. The case $\varphi(\tau) \rightarrow 0$ as $\tau \rightarrow +\infty$ is always regular. Indeed, in this case, as follows from (B.20), $(0, 0)$ is always regular, since $\varphi(\tau) \rightarrow 0$ and hence, by just the continuity of the solution $v(y, \tau)$, we have $v(0, \tau) \rightarrow 0$ as $\tau \rightarrow +\infty$. It is worth mentioning that this is a completely *rigorous* result. Therefore, according to our blow-up asymptotic approach, the regularity problem of the vertex $(0, 0)$ is trivial. Note that it is not that trivial via general PDE approaches (not including blow-up scalings); cf. much weaker assumptions in, e.g., [12].

B.7. Second scaling: Boundary Layer (BL) structure. Meantime, we return to the case of unbounded functions $\varphi(\tau)$'s. Then, using standard boundary layer concepts of Prandtl–Blasius developed in 1904–08, we observe that, sufficiently close to the right-hand lateral boundary of Q_0 , it is natural to introduces the variables

$$(B.25) \quad z = \frac{y}{\varphi(\tau)} \quad \text{and} \quad v(y, \tau) = w(z, \tau) \quad \implies \quad w_\tau = \frac{1}{\varphi^2} i w_{zz} - \frac{1}{2} z w_z + \frac{\varphi'}{\varphi} z w_z.$$

We next introduce the standard BL-variables (the same as for the heat equation)

$$(B.26) \quad \xi = \varphi^2(\tau)(1 - z) \equiv \varphi(y - \varphi), \quad \varphi^2(\tau) d\tau = ds, \quad \text{and} \quad w(z, \tau) = \rho(s)g(\xi, s),$$

where $\rho(s)$ is an unknown slow decaying (in the same natural sense, associated with (B.18)) time-factor depending on the function $\varphi(\tau)$. We will use later on the fact that

$$(B.27) \quad \rho = a_0(\tau)(1 + o(1)) \quad \text{as} \quad \tau \rightarrow +\infty,$$

where $a_0(\tau)$ is the first Fourier coefficient of the solution $v(y, \tau)$ relative to the adjoint basis $\{\psi_k^*\}$ of the operator \mathbb{B}^* .

On substitution into the PDE in (B.25), we obtain the following perturbed equation (see details of a similar derivation in [25, § 7.2]):

$$(B.28) \quad g_s = \mathbf{A}g - \frac{1}{\varphi^2} \left(\frac{1}{2} + \frac{\varphi'}{\varphi} \right) \xi g_\xi - \frac{\varphi'}{\varphi} g_\xi, \quad \text{where} \quad \mathbf{A}g = i g'' + \frac{1}{2} g'.$$

As usual in boundary layer theory, we are looking for a generic pattern of the behaviour described by (B.28) on compact subsets near the lateral boundary,

$$(B.29) \quad |\xi| = o(\varphi^{-2}(\tau)) \quad \implies \quad |z - 1| = o(\varphi^{-4}(\tau)) \quad \text{as} \quad \tau \rightarrow +\infty.$$

On these space-time compact subsets, the second term on the right-hand side of (B.28) becomes asymptotically small, while all the others are much smaller in view of the slow growth/decay assumptions such as (B.18) for $\varphi(\tau)$ and $\rho(s)$.

Then posing the asymptotic behaviour at infinity: $g(\xi, \tau)$ is bounded as $\xi \rightarrow +\infty$, and

$$(B.30) \quad \sup_{\xi} |g(\xi, s)| = 1 \quad (\text{Hypothesis I for generic patterns}).$$

This is a typical “normalization by 1” condition from boundary layer theory. Note that, in view of highly oscillatory nature of any solutions of Schrödinger equations, we cannot normalize by 1 any component of $g(\xi, s) = h(\xi, s) + i w(\xi, s)$. The condition (B.30) will be used for matching with the solution asymptotics in the Inner Region.

Then, as $\xi \rightarrow +\infty$, all the derivatives are assumed to be bounded, and we arrive at a standard stabilization issue of passing to the limit as $s \rightarrow +\infty$ in (B.28), (B.30). Assuming that, by the definition in (B.26), the rescaled orbit $\{g(s), s > 0\}$ is uniformly bounded, by classic parabolic theory [18], one can pass to the limit in (B.28) along a subsequence $\{s_k\} \rightarrow +\infty$. Namely, by the above, we have that, uniformly on compact subsets defined in (B.29), as $k \rightarrow \infty$,

$$(B.31) \quad g(s_k + s) \rightarrow h(s), \quad \text{where} \quad h_s = \mathbf{A}h, \quad h = 0 \quad \text{at} \quad \xi = 0,$$

and (cf. (B.30)) h is bounded (i.e., being oscillatory) as $\xi \rightarrow +\infty$. The *limit* (at $s = +\infty$) equation obtained from (B.28):

$$(B.32) \quad h_s = \mathbf{A}h \equiv i h_{\xi\xi} + \frac{1}{2} h_{\xi}$$

is a standard linear PDE in the unbounded domain \mathbb{R}_+ , though it is governed by a non self-adjoint operator \mathbf{A} . We then need the following its property: in an appropriate weighted L^2 -space if necessary and under the hypothesis (B.30), the stabilization holds, i.e., the ω -limit set of $\{h(s)\}$ consists of equilibria: as $s \rightarrow +\infty$,

$$(B.33) \quad \begin{cases} h(\xi, s) \rightarrow g_0(\xi), & \text{where} \quad \mathbf{A}g_0 = 0 \quad \text{for} \quad \xi > 0, \\ g_0 = 0 & \text{for} \quad \xi = 0, \quad \sup_{\xi} |g_0(\xi)| = 1 \quad (\text{bounded in } \mathbb{R}_+). \end{cases}$$

The stationary problem in (B.33) can be easily solved to give the BL profile

$$(B.34) \quad g_0(\xi) = \frac{1}{2} (1 - e^{i \frac{\xi}{2}}), \quad \text{where} \quad |g_0(\xi)| = |\cos(\frac{\xi}{4})| \leq 1.$$

We must admit that (B.33) and (B.34) *actually define* the class of solutions we are going to treat later on. Hopefully, this should be a generic class. So, we will not concentrate on the stabilization problem (B.33), which reduces to a standard spectral study of \mathbf{A} in a weighted space. Actually, the convergence (B.31) and (B.33) for the perturbed dynamical system (B.28) is the main HYPOTHESIS (H), which characterizes the class of generic patterns under consideration, and then (B.30) is its partial consequence. Note that the uniform stability of the stationary point g_0 in the limit autonomous system (B.32) in a suitable metric will guarantee that the asymptotically small perturbations do not affect the omega-limit set; see [29, Ch. 1]. Such a definition of generic patterns looks rather non-constructive, which, however, is unavoidable for higher-order PDEs without positivity and order-preserving issues.

B.8. Inner Region expansion: towards regularity. We next proceed as in [25, § 7.3, 7.7]. Namely, in Inner Region, we deal with the original rescaled problem (B.20).

In order to apply the standard eigenfunction expansion techniques by using the orthonormal set of polynomial eigenfunctions of \mathbb{B}^* given in (4.16), as customary in classic PDE and potential theory, we extend $v(y, \tau)$ by 0 for $y > \varphi(\tau)$ by setting:

$$(B.35) \quad \hat{v}(y, \tau) = v(y, \tau)H(\varphi(\tau) - y) = \begin{cases} v(y, \tau) & \text{for } 0 \leq y < \varphi(\tau), \\ 0 & \text{for } y \geq \varphi(\tau), \end{cases}$$

where H is the Heaviside function. Since $v = 0$ on the lateral boundary $\{y = \varphi(\tau)\}$, in the sense of distributions,

$$(B.36) \quad \hat{v}_\tau = v_\tau H, \quad \hat{v}_y = v_y H, \quad \hat{v}_{yy} = v_{yy} H - v_y|_{y=\varphi} \delta(y - \varphi).$$

Therefore, \hat{v} satisfies the following equation:

$$(B.37) \quad \hat{v}_\tau = \mathbf{B}^* \hat{v} + v_y|_{y=\varphi} \delta(y - \varphi) \quad \text{in } \mathbb{R}_+ \times \mathbb{R}_+.$$

Since the extended solution orbit (B.35) is assumed to be uniformly bounded in $L^2_{\rho^*}(\mathbb{R})$, we use the converging in the mean (and uniformly on compact subsets in y) the eigenfunction expansion via the generalized Hermite polynomials (4.16):

$$(B.38) \quad \hat{v}(y, \tau) = \sum_{(k \geq 0)} a_k(\tau) \psi_k^*(y).$$

Obviously, this assumes the inclusion $v(\cdot, \tau) \in \tilde{L}^2_\rho$, for all $\tau \gg 1$, which, by classic regularity theory for Schrödinger equations, is not a restrictive assumption at all. Then, substituting (B.38) into (B.37) and using the orthonormality property (D.7) in a “*v.p.*” sense of (D.6) yields the following dynamical system for the expansion coefficients:

$$(B.39) \quad a'_k = \lambda_k a_k + v_y|_{y=\varphi(\tau)} \langle \delta(y - \varphi(\tau)), \psi_k \rangle \quad \text{for all } k = 0, 1, 2, \dots,$$

where $\lambda_k = -\frac{k}{2}$ are real eigenvalues (3.33) of \mathbb{B}^* . Recall that $\lambda_k < 0$ for all $k \geq 1$. More importantly, the corresponding eigenfunctions $\psi_k(y)$ are unbounded and not monotone for $k \geq 1$ according to (4.16). Therefore, regardless proper asymptotics given by (B.39), these inner patterns cannot be matched with the BL-behaviour such as (B.30), and demand other matching theory. Since these are not generic, the latter will be dropped, though can be taken into account for a full classification of (non-generic) asymptotics.

Thus, one needs to concentrate on the “maximal” first Fourier generic pattern associated with

$$(B.40) \quad k = 0 : \quad \lambda_0 = 0 \quad \text{and} \quad \psi_0^*(y) \equiv 1 \quad (\psi_0(y) = F(y)),$$

where $F(y)$ is the “Gaussian” profile (2.6). Actually, this corresponds to a naturally understood “centre subspace behaviour” for the equation (B.39):

$$(B.41) \quad \hat{v}(y, \tau) = a_0(\tau) \cdot 1 + w^\perp(y, \tau), \quad \text{where } w^\perp \in \text{Span}\{\psi_k^*, k \geq 1\},$$

and $w^\perp(y, \tau)$ is then negligible relative to $a_0(\tau)$. This is another characterization of our class of generic patterns, Hypothesis II. The equation for $a_0(\tau)$ then takes the form:

$$(B.42) \quad a'_0 = v_y|_{y=\varphi(\tau)} \psi_0(\varphi(\tau)).$$

We now return to BL theory established the boundary behaviour (B.26) for $\tau \gg 1$, which for convenience we state again: in the rescaled sense, on the given compact subsets,

$$(B.43) \quad v(y, \tau) = \rho(s) g_0(\varphi^2(\tau)(1 - \frac{y}{\varphi(\tau)})) + \dots$$

Of course, since the limit BL-profile (B.34) is uniformly bounded but essentially oscillatory, we are talking about matching of this BL-asymptotics with the constant one in (B.41) *only* in a

natural “average sense”. E.g., after a standard integral averaging the oscillatory BL-profile, which eliminates the non-essential multiplier $\frac{1}{2}$ in (B.34), since⁷

$$\frac{1}{L} \int_0^L \cos^2\left(\frac{\xi}{4}\right) d\xi \rightarrow \frac{1}{2} \quad \text{as } L \rightarrow +\infty.$$

Note that there are (finitely) oscillatory generalized Hermite polynomials $\psi_k^*(y) \equiv H_k(iy)$ for large k (see (4.16), $k = \beta$ in 1D), but these all are *unbounded* as $y \rightarrow \infty$, so cannot be matched with uniformly bounded BL-expansions.

By the matching of both Regions, one concludes that, for such generic patterns,

$$(B.44) \quad \frac{a_0(\tau)}{\rho(s)} \rightarrow 1 \quad \text{as } \tau \rightarrow +\infty \implies \rho(s) = a_0(\tau)(1 + o(1)).$$

Then the convergence (B.31), which by a standard regularity is assumed to be also true for the spatial derivatives, yields, in the natural rescaled sense, that, as $\tau \rightarrow +\infty$,

$$(B.45) \quad v_y|_{y=\varphi(\tau)} \rightarrow \rho(s)\varphi(\tau)\gamma_1 \rightarrow a_0(\tau)\varphi(\tau)\gamma_1, \quad \text{where } \gamma_1 = -g'_0(0) = \frac{i}{2}.$$

We again recall that such an estimate is assumed to be true for a fixed above generic class of solutions satisfying a proper stabilization property in the BL.

Thus, this leads to an asymptotic ODE for the first expansion coefficient for generic patterns:

$$(B.46) \quad a'_0 = \hat{\gamma}_1 a_0 \varphi(\tau) e^{i \frac{|\varphi(\tau)|^2}{4}} + \dots, \quad \text{where } \hat{\gamma}_1 = \frac{i}{2\sqrt{4\pi i}} = \frac{1-i}{4\sqrt{2\pi}}.$$

This not-that-easy asymptotic ODE gives insight into main difficulties that one can face while posing and studying the problem on the boundary regularity of the vertex $(0, 0)$ for Schrödinger-type operators. To this end, we first derive the real form of this system for

$$(B.47) \quad a_0(\tau) = b_0(\tau) + i d_0(\tau),$$

where these parts now satisfy the system:

$$(B.48) \quad \begin{cases} b'_0 = \frac{\varphi(\tau)}{4\sqrt{2\pi}} \left[(b_0 + d_0) \cos\left(\frac{\varphi^2(\tau)}{4}\right) + (b_0 - d_0) \sin\left(\frac{\varphi^2(\tau)}{4}\right) \right] + \dots, \\ d'_0 = \frac{\varphi(\tau)}{4\sqrt{2\pi}} \left[(-b_0 + d_0) \cos\left(\frac{\varphi^2(\tau)}{4}\right) + (b_0 + d_0) \sin\left(\frac{\varphi^2(\tau)}{4}\right) \right] + \dots \end{cases}$$

Note that the regularity of $(0, 0)$ assumes that both limits are zero:

$$(B.49) \quad b_0(\tau) \quad \text{and} \quad d_0(\tau) \rightarrow 0 \quad \text{as } \tau \rightarrow +\infty.$$

The system (B.48) shows a general range of problems that appear determining the conditions on the lateral boundary given by the function $\varphi(\tau)$ to ensure (B.49). Of course, there is no any hope to guarantee (B.49) via a kind of Osgood (1898)–Dini-like integral condition of Petrovskii’s type for the heat equation (B.8). The latter one has a very simple form (see a derivation in the same lines of spectral properties of \mathbf{B}^* in [25, § 7.7]):

$$(B.50) \quad \boxed{(0, 0) \text{ is regular iff } \int_0^\infty \varphi(s) e^{-\frac{\varphi^2(s)}{4}} ds = +\infty,}$$

which was already obtained by Petrovskii in 1934 [75, 76] (earlier related results were due to Khinchin, 1924, in a probability representation; see [25, § 3.2] for further details).

As a clue to some hard features of the system (B.48), consider as a “toy-model” a single equation of a similar form:

$$(B.51) \quad b'_0 \sim \frac{\varphi(\tau)}{4\sqrt{2\pi}} b_0 \cos\left(\frac{\varphi^2(\tau)}{4}\right).$$

⁷This is about averaging of $|g_0|^2$; averaging of $|g_0|$ will give another (non-important) constant.

Integrating one obtains that

$$(B.52) \quad \ln |b_0(\tau)| \sim \frac{1}{4\sqrt{2\pi}} \int_0^\tau \varphi(s) \cos\left(\frac{\varphi^2(s)}{4}\right) ds.$$

In particular, the regularity of $(0,0)$ demands that

$$(B.53) \quad \ln |b_0(\tau)| \rightarrow -\infty \implies \int_0^\tau \varphi(s) \cos\left(\frac{\varphi^2(s)}{4}\right) ds \rightarrow -\infty \quad \text{as } \tau \rightarrow +\infty.$$

This immediately implies (this is true for the toy-model, but it seems such a condition exists for the whole system (B.48))

$$(B.54) \quad \int_0^\tau \varphi(s) \cos\left(\frac{\varphi^2(s)}{4}\right) ds \text{ converges} \implies (0,0) \text{ is irregular.}$$

It is easy to see that for the power-type functions

$$(B.55) \quad \varphi(\tau) \sim \tau^\alpha \implies \alpha > 1 \text{ means integral convergence and hence irregularity.}$$

Though $\varphi(\tau)$ in (B.55) is not a slow growing function, the above results are extended to that case. Hence, for any slow growing functions, the integral in (B.54) always diverges.

However, on the other hand, the divergence of the integral in (B.54), e.g., for $\alpha \leq 1$ in (B.55) *does not imply any regularity*. Indeed, according to (B.53), the divergence must be to $-\infty$, so that to create a regular boundary point $(0,0)$ a special *oscillatory cut-off of the boundary* is necessary. This is such a “correction” of the shape of the boundary that eliminates the positive divergence part in the integrals in (B.53). Such a procedure, which establishes a coherent behaviour of the lateral boundary close to the characteristic point $(0,0)$ and oscillations of the rescaled kernel $F(y)$, is inevitable for infinitely oscillatory fundamental solutions. This oscillatory cut-off already appears for the bi-harmonic equation (1.9); see [25, § 7.5], where further related details can be found.

We hope that the above analysis correctly describes a full range of problems that appear in the regularity analysis and also correctly shows how $\{\mathbb{B}, \mathbb{B}^*\}$ -spectral theory naturally enters this classic PDE area.

APPENDIX C. Application V: TOWARDS COUNTABLE FAMILIES OF NONLINEAR EIGENFUNCTIONS OF THE QLSE

C.1. The QLSE and its applications. This is a more striking and even controversial application of the spectral $\{\mathbb{B}, \mathbb{B}^*\}$ theory developed above. Namely, we now turn to a $2m$ th-order *quasilinear Schrödinger equation* (the QLSE- $2m$) (1.16). Quasilinear Schrödinger-type models, some of which can be written as

$$(C.1) \quad i u_t + \Delta u + \beta |u|^{p-1} u + \theta (\Delta |u|^2) u = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+,$$

are not a novelty in several physical situations such as superfluid theory, dissipative quantum mechanics, and in turbulence theory. We refer to some papers from the 1970s and 1980s [38, 79, 84], to [34] for further references, and to [63] for a more mathematical knowledge, as well as to [64] and to Zakharov’s *et al* papers [96, 97], as a sufficient source of other reference and deep physical results. Note that the quasilinear model proposed in [64] in 1997 is more related to a kind of a “ p -Laplacian operator” structure with fractional derivatives, such as

$$(C.2) \quad i u_t = \lambda |D_x|^{\frac{\beta}{4}} (| |D_x|^{\frac{\beta}{4}} u |^2 |D_x|^{\frac{\beta}{4}} u) + |D_x|^\alpha u,$$

with a standard Fourier-definition of operators $|D_x|^\alpha$, having the symbol $|\xi|^\alpha$, so that, in particular $|D_x|^2 = -D_x^2 > 0$. Then $\lambda = 1$ in (C.2) corresponds to the original defocusing model. Here, real parameters α and β control dispersion and nonlinearity respectively. The standard NLSE then occurs for $\alpha = 2$ and $\beta = 0$ in (C.2).

For $n = 0$, the QLSE (1.16) formally reads as the linear original one (1.1). Indeed, the QSLE (1.16) still remains rather rare and seems even an exotic equation in PDE theory, and therefore we now are going to propose a general approach to understanding of its new internal properties.⁸

Overall, bearing in mind some clear discrepancies between the physical quasilinear models and the proposed one (1.16), we just say that this one was chosen as a typical example only to demonstrate our branching approach, so that others, more physically motivated models, would do the same, when the main mathematical ideas would have been properly explained and motivated.

C.2. Two “adjoint” nonlinear eigenvalue problems. Thus, as in the linear case for $n = 0$, we are going to study *global asymptotic behaviour* (as $t \rightarrow +\infty$) and *finite-time blow-up behaviour* (as $t \rightarrow T^- < +\infty$) of solutions of the QLSE (1.16).

Overall, we are looking for similarity solutions of (1.16) of two “forward” and Sturm’s “backward” types:

- (i) *global similarity patterns* for $t \gg 1$, and
- (ii) *blow-up similarity ones* with the finite-time behaviour as $t \rightarrow T^- < \infty$.

Both classes of such particular solutions of the QSLE (1.16) are written in the joint form as follows, by setting $T = 0$ in (ii):

$$(C.3) \quad u_\pm(x, t) = (\pm t)^{-\alpha} f(y), \quad y = x/(\pm t)^\beta, \quad \text{where } \beta = \frac{1-\alpha n}{2m}, \quad \text{for } \pm t > 0,$$

where similarity profiles $f(y)$ satisfy the following *nonlinear eigenvalue problems*, respectively,

$$(C.4) \quad \boxed{(\mathbf{NEP})_\pm : \quad \mathbf{B}_n^\pm(\alpha, f) \equiv -i(-\Delta)^m(|f|^n f) \pm \beta y \cdot \nabla f \pm \alpha f = 0 \quad \text{in } \mathbb{R}^N.}$$

Here, $\alpha \in \mathbb{R}$ is a parameter, which stands in both cases for admitted *real (!) nonlinear eigenvalues*. Thus, the sign “+”, i.e., $t > 0$, corresponds to global asymptotics as $t \rightarrow +\infty$, while “−” ($t < 0$) yields blow-up limits $t \rightarrow T = 0^-$ describing a “micro-scale” structures of the PDE. In fact, the blow-up patterns are assumed to describe the structures of “multiple zeros” of solutions of the QLSE. As we have mentioned, this idea goes back to Sturm’s analysis of solutions of the 1D heat equation performed in 1836 [85]; see [23, Ch. 1] for the whole history and applications of these fundamental Sturm’s ideas and two *zero set Theorems*.

Being equipped with proper “boundary conditions at infinity”, namely,

$$(C.5) \quad \text{for global case, } \mathbf{B}_n^+(\alpha, f) : \quad f(y) \text{ is “maximally” oscillatory as } y \rightarrow \infty, \quad \text{and}$$

$$(C.6) \quad \text{for blow-up case, } \mathbf{B}_n^-(\alpha, f) : \quad f(y) \text{ has a “minimal growth” as } y \rightarrow \infty,$$

equations (C.4) produce *true two nonlinear eigenvalue problems* to study, which can be considered as a pair of mutually “adjoint” ones. Note that (C.6) actually also means that the admitted

⁸While being in Bath in 2008, Peter Markowitz, answering a question of the first author, fast and witty called (1.16), $m = 1$, an *NLSE of the porous medium type* bearing in mind the classic parabolic PME

$$u_t = \Delta(|u|^n u) \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad \text{where } n > 0.$$

nonlinear eigenfunctions *are not* of a type of a maximal oscillatory behaviour at infinity that connects us with the issue (4.13) (which, however, is not sufficient, and a growth analysis at infinity should be involved in parallel, as shown below).

Let us discuss in greater detail the meaning of those above conditions at infinity. Firstly, (C.5) means that, due to the type of nonlinearity $|f|^n$, the oscillatory component such as (2.11) is admissible, with, of course, an extra generated algebraic factor of the WKBJ-type, which we do not specify hereby. This can be explained as follows: if $f(y)$ has a standard WKBJ-type two-scale asymptotics

$$(C.7) \quad f(y) \sim |y|^\delta e^{a|y|^\alpha} \quad \text{as } y \rightarrow \infty,$$

then, since $|e^{a|y|^\alpha}| = 1$ for $a \in i\mathbb{R}$, substituting into (C.4)₊ yields the balance

$$(C.8) \quad \delta n + (2m - 1)(\alpha - 1) = 1,$$

i.e., different from the purely linear one as in (2.9), since, for $n > 0$, the exponent δ from the slower varying factor is involved (we do not calculate it here being a standard asymptotic procedure).

Secondly, (C.6) assumes actually also a “minimal” growth at infinity. Namely, quite similar to the linear problem for $n = 0$, the first two terms in (C.4)₋ generate a fast growing bundle: as $y \rightarrow \infty$ (as usual, we omit slower oscillatory components)

$$(C.9) \quad -i(-\Delta)^m(|f|^n f) - \beta y \cdot \nabla f + \dots = 0 \quad \implies \quad f(y) \sim |y|^{\frac{2m}{n}}.$$

On the other hand, two linear terms in (C.4)₋ lead to a different slower growth as $y \rightarrow \infty$:

$$(C.10) \quad \dots - \beta y \cdot \nabla f - \alpha f = 0 \quad \implies \quad f(y) \sim |y|^{-\frac{\alpha}{\beta}} \equiv |y|^{\frac{2m|\alpha|}{1+|\alpha|n}}$$

(recall that $\alpha(0) = \lambda < 0$). Since

$$(C.11) \quad \frac{2m|\alpha|}{1+|\alpha|n} < \frac{2m}{n},$$

this actually means that (C.6) establishes a kind of a “minimal” growth of admissible nonlinear eigenfunctions at infinity corresponding to (C.10). For $n = 0$, this implies a *polynomial* growth, and all the admissible (extended) eigenfunctions of \mathbb{B}^* turned out to be generalized Hermite polynomials (4.16). Note that, in self-similar approaches and ODE theory, such “minimal growth” conditions are known to define similarity solutions of the *second kind*, a term, which was introduced by Ya.B. Zel’dovich in 1956 [98], and many (but indeed easier) such ODE problems have been rigorously solved since that. For quasilinear problems such as (C.4), the condition (C.6) is incredibly more difficult. We thus cannot somehow rigorously justify that the problem (C.4)₋, (C.6) is well posed and admits a countable family of solutions and nonlinear eigenvalues $\{\alpha_\gamma^-(n)\}$. Actually, the homotopy deformation as $n \rightarrow 0^+$ is the only our original intention *to avoid* such a difficult “direct” mathematical study of this nonlinear blow-up eigenvalue problem.

All related aspects and notions used above and remaining unclear will be properly discussed and specified.

Of course, these conditions (C.5) and (C.6) remind us the “linear” ones associated with (2.13) for \mathbb{B} and (4.13) for \mathbb{B}^* respectively, justified earlier for $n = 0$. Indeed, a better understanding of those conditions in the nonlinear case $n > 0$ demands a much more difficult mathematics. However, one can observe that both (C.5) and (C.6) are just two *asymptotic* (not global ones) problems concerning admitted behaviour of solutions of (C.4) as $y \rightarrow \infty$, so that, at this moment we are in a position to neglect these and to face more fundamental issues to be addressed below.

Note also that, at least, in 1D or for radially symmetric solutions in \mathbb{R}^N , such asymptotic problems for not that hard nonlinear ODEs are easily solvable.

Thus, for $n = 0$, equations (C.4), equipped with proper weighted L^2 spaces, take the very familiar form: the corresponding differential expressions are

$$(C.12) \quad \mathbf{B}, \mathbf{B}^* : \quad -i(-\Delta)^m f \pm \beta y \cdot \nabla f \pm \alpha f = 0 \quad \text{in } \mathbb{R}^N.$$

Then we observe the obvious relation between α 's and λ 's from the spectrum $\sigma(\mathbb{B}) = \sigma(\mathbb{B}^*)$:

$$(C.13) \quad \text{for } \mathbb{B} : \quad \alpha = -\lambda + \frac{N}{2m} \quad \text{and} \quad \text{for } \mathbb{B}^* : \quad \alpha = \lambda.$$

Thus, our next goal is to show, by using any means, that, at least for small $n > 0$, the nonlinear eigenvalue problems

$$(C.14) \quad (\mathbf{NEP})_{\pm} \text{ admit countable sets of solutions } \Phi^{\pm}(n) = \{\alpha_{\gamma}^{\pm}, f_{\gamma}^{\pm}\}_{|\gamma| \geq 0},$$

where, as usual and as it used to be in the linear case, γ is a multiindex in \mathbb{R}^N to numerate the pairs.

The last question to address is whether these sets

$$(C.15) \quad \Phi^{\pm}(n) \text{ of nonlinear eigenfunctions are } \textit{evolutionary complete},$$

i.e., describe *all* possible asymptotics as $t \rightarrow +\infty$ and $t \rightarrow 0^-$ (on the corresponding compact subsets in the variable y in (C.3)) in the CP for the QLSE (1.16) with bounded integrable (and possibly compactly supported, – any assumption is allowed) initial data. Our main approach is the idea of a “homotopic deformation” of (1.16) as $n \rightarrow 0^+$ and reducing it to our linear equation (1.1), for which both problems (C.14) and (C.15) are solved positively by a non-standard and not self-adjoint spectral theory of the linear operator pair $\{\mathbb{B}, \mathbb{B}^*\}$.

C.3. Example: first explicit nonlinear eigenfunctions. For $m = 1$, the problem (C.4)₊ has the following first pair associated with the explicit kernel (2.6):

$$(C.16) \quad \alpha_0^+(n) = \frac{N}{2+Nn} \quad \text{and} \quad f_0^+(y) = \left(\frac{2}{2+Nn}\right)^{\frac{1}{n}} e^{\frac{i|y|^2}{4}}.$$

Not that surprisingly, regardless the degeneracy of the QLSE (1.16) at the zero level $\{u = 0\}$, (C.16) shows that the solution exhibits no *finite interfaces*. This is in a striking difference with, say, as a typical example, the TFE–4

$$(C.17) \quad u_t = -\nabla \cdot (|f|^n \nabla \Delta f) \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad \text{where } n > 0,$$

which is known to admit compactly supported solutions in both the FBP and the Cauchy problem setting; see [30, 31] and, respectively, [19, 20], as a source of main results and further references.

Concerning the “blow-up problem”, for any $m \geq 1$, the adjoint nonlinear eigenvalue one (C.4)_– has the obvious first pair

$$(C.18) \quad \alpha_0^-(n) = 0 \quad \text{and} \quad f_0^-(y) \equiv 1,$$

i.e., the same as for $n = 0$, where $\psi_0^*(y) \equiv 1$, as the first Hermite polynomial; see (4.16).

Anyway, in our further analysis, we cannot rely on any explicit representation of any nonlinear eigenfunctions. So, we now very briefly begin to explain our approach.

C.4. Branching for the “forward” problem for $t \gg 1$. We perform our bifurcation-branching analysis following the lines of classic theory [7, 15, 58, 90], etc., in the case of *finite* regularity. However, we must admit that this classic one does not cover rigorously the type of n -branching as $n \rightarrow 0$, which requires extra difficult study, which we cannot address here.

Thus, we are looking for a countable set of nonlinear pairs $\{a_\gamma^+(n), f^+(y; n)\}_{|\gamma| \geq 0}$, which are assumed to describe *all* (hopefully, for $n > 0$ small) asymptotic patterns for the QLSE (1.16) as $t \rightarrow +\infty$, *up to possible centre manifold patterns*; see Section 5.2. However, as we have seen, such special patterns do occur, if there is a certain *interaction* (a *transitional behaviour*) between the linear and the nonlinear terms in equations such as (5.7). This looks rather unreal for the current problem under consideration. However, the *evolutionary completeness* of the nonlinear patterns $\Phi^+(n)$ as $t \rightarrow +\infty$ remains an extremely difficult open problem, which probably will be extremely hard to solve completely rigorously.

Thus, assuming that $n > 0$ is small, we perform asymptotic expansions in the operators and coefficients in $(C.4)_+$. Evidently, the crucial one is in the nonlinearity, which requires:

$$(C.19) \quad |f|^n f \equiv f e^{n \ln |f|} = f(1 + n \ln |f| + o(n)) \quad \text{as } n \rightarrow 0^+.$$

It is clear that the neighbourhoods of the nodal set of $f(y)$ are key for (C.19) to be valid in any weak sense (precisely this is needed for the equivalent analysis of the inverse *integral* compact operators involved, where a proper justification must take place). If $f(y)$ has a *nice nodal set consisting of a.a. isolated and “transversal” a.e. zero surfaces (or just some points only), with no “thin” concentration subsets*, the expansion (C.19) can be valid even in the standard pointwise sense, or at least in the weak sense. However, we do not know and currently cannot prove such deep properties of the nonlinear eigenfunctions involved. Note that, for $n = 0$, the generating formula (3.34), with a proper knowledge of such nice zero set properties of the rescaled fundamental kernel $F(y)$ (this is doable), guarantees such necessary properties of eigenfunctions $\psi_\gamma(y)$.

The above discussion establishes the main hypothesis to make our branching analysis to be (almost) rigorous, being applied, of course, to the equivalent integral equation, where establishing some further compact and other necessary properties of the nonlinear integral operators would take some time, indeed. This can be also done, since the problems $(C.4)_\pm$ can be reduced to a semilinear form.

The rest of the expansions in $(C.4)_+$ are straightforward (here, we already fix by γ an n -branch we are going to trace out):

$$(C.20) \quad \begin{aligned} \alpha_\gamma(n) &= \alpha_\gamma(0) + \hat{\alpha}_\gamma n + \dots \equiv -\lambda_\gamma + \frac{N}{2m} + \hat{\alpha}_\gamma n + \dots, \\ \frac{1 - \alpha_\gamma(n)n}{2m} &= \frac{1}{2m} - a_\gamma n + \dots, \quad \text{where } a_\gamma = \frac{-\lambda_\gamma + \frac{N}{2m}}{2m}, \end{aligned}$$

where we have already omitted all $o(n)$ -terms (this again assumes extra regularity hypothesis already discussed above). In the first line in (C.20), the parameter $\hat{\alpha}_\gamma$ is an extra unknown.

Substituting all the expansions into $(C.4)_+$ yields the following perturbed problem:

$$(C.21) \quad (\mathbb{B} - \lambda_\gamma I)f + n h + \dots = 0, \quad \text{where } h = [\mathbf{i}(-1)^{m+1} \Delta^m(f \ln |f|) - a_\gamma y \cdot \nabla f + \hat{\alpha}_\gamma f].$$

Thus, as $n \rightarrow 0^+$, we must look for a solution close to the eigenspace

$$(C.22) \quad \ker(\mathbb{B} - \lambda_\gamma I) = \text{Span}\{\psi_\gamma, |\gamma| = l\}.$$

Therefore, under prescribed hypothesis, solutions take the form

$$(C.23) \quad f = \sum_{|\sigma|=l} c_\sigma \psi_\sigma + n\phi_\gamma + \dots,$$

where the expansions coefficients $\{c_\sigma\}_{|\sigma|=l}$ and the orthogonal part ϕ_γ are unknowns.

Finally, substituting (C.23) into (C.21) yields the $O(n)$ -problem

$$(C.24) \quad (\mathbb{B} - \lambda_\gamma I)\phi_\gamma + h = 0.$$

Thus, the necessary (and, in properly regular cases, the sufficient) orthogonality condition of the solvability of (C.24),

$$(C.25) \quad h \perp^* \ker(\mathbb{B} - \lambda_\gamma I),$$

yields the following Lyapunov–Schmidt scalar branching equation: for all $|\delta| = l$,

$$(C.26) \quad \langle i(-1)^{m+1} \Delta^m (\sum c_\sigma \psi_\sigma \ln |\sum c_\sigma \psi_\sigma|) - a_\gamma y \cdot \nabla \sum c_\sigma \psi_\sigma + \hat{a}_\gamma \sum c_\sigma \psi_\sigma, \psi_\delta^* \rangle_* = 0,$$

where an integration by parts in the second term can be performed to simplify the expressions; see the next Appendix D for further details concerning the meaning of such extended generalized linear functionals. We must admit again that a proper well posed way for calculating values of such extended functionals is not available, so we present such a branching analysis just as an example reminding analogies with the classic approaches.

Overall, (C.26) is the required *algebraic system* for the unknowns \hat{a}_γ and $\{c_\sigma\}$ (a convenient normalization condition on the latter expansion coefficients may be added). As usual, once the branching equation (C.26) has been properly solved, this allows one to get the corresponding unique solution of the differential equation (C.24), etc.

Indeed, (C.26) is a very difficult algebraic system, which is not of any variational form, so one cannot use powerful category-genus theory [7, 58] to predict a number of solutions, i.e., a number of such n -branches originated from the given eigenspace in (C.22). Note that, for $m = 1$, when there exists some “symmetry” of differential forms \mathbf{B} , \mathbf{B}^* reflected in (3.8) and (3.27), the system (C.26) reveals some “variational-like” features, since the eigenfunctions ψ_γ and polynomials ψ_γ^* can be identified in a weighted space. However, this is supposed to happen in a space with indefinite metric, so we do not check how this can be helpful.

It is worth mentioning that the problem on a sharp estimate of a number of solution branches emanating from an eigenspace, remains essentially open even for classes of well-understood variational operators. On one hand, the first conclusion is classic: the number of branches is not less than the dimension of the eigenspace: indeed, since the corresponding algebraic system (like (C.26)) remains also variational, the category of the functional set is not less than the linear eigenspace dimension, whence the result. But, obviously, a sharper estimate of the solutions number becomes essentially nonlinearity-dependent, so that this is not (and, possibly, cannot be in the maximal generality) completely understood. For a number of branches that can emanate from the trivial solution, there have been obtained some specific examples only. There exist some results for potential operators (see [14] and [80] as a guide), and a very few for non-gradient and non-self-adjoint operators [61]. For the TFE-4 (C.17), such a branching analysis as $n \rightarrow 0^+$ reveals a lot of technical difficulties, though some problems for simple and semisimple eigenvalues are shown to admit a rather definite path towards, [2].

C.5. Blow-up scaling problem: $t \rightarrow 0^-$. This is quite similar. Since the nonlinear eigenvalue equations (C.4) $_{\pm}$ differ by the sign in the linear terms only (but the boundary-radiation conditions at infinity are entirely different), instead of (C.21), we arrive at

$$(C.27) \quad (\mathbb{B}^* - \lambda_{\gamma} I) f + n h + \dots = 0, \quad \text{where } h = [\mathbf{i}(-1)^{m+1} \Delta^m (f \ln |f|) + a_{\gamma} y \cdot \nabla f - \hat{\alpha}_{\gamma} f].$$

Therefore, looking for solutions (C.23), now over the kernel of the adjoint operator $(\mathbb{B}^* - \lambda_{\gamma} I)$,

$$(C.28) \quad f = \sum_{|\sigma|=l} c_{\sigma} \psi_{\sigma}^* + n \phi_{\gamma}^* + \dots,$$

we arrive at the “adjoint” algebraic system: for all $|\delta| = l$,

$$(C.29) \quad \langle \mathbf{i}(-1)^{m+1} \Delta^m (\sum c_{\sigma} \psi_{\sigma}^* \ln |\sum c_{\sigma} \psi_{\sigma}^*|) + a_{\gamma} y \cdot \nabla \sum c_{\sigma} \psi_{\sigma}^* - \hat{\alpha}_{\gamma} \sum c_{\sigma} \psi_{\sigma}^*, \psi_{\delta} \rangle_* = 0,$$

which are not easier than the first one (C.26), and does not have any variational structure, so the same principal difficulties on the solvability (this is easier to do in the 1D or the radial case) occurs, and especially on the number of solutions.

Recall that the patterns

$$(C.30) \quad u(x, t) \sim e^{\alpha_{\gamma}(n)\tau} \sum_{|\sigma|=l} c_{\sigma} \psi_{\sigma}^*(y) + \dots, \quad \text{where } y = \frac{x}{(-t)^{\beta_{\gamma}(n)}}, \quad \tau = -\ln(-t),$$

are assumed to describe, for all finite multiindices γ and all admitted solutions of the branching equation (C.29), the whole variety of “micro-scale patterns”, which are available for the QLSE (1.16) at a given arbitrary point $(0, 0^-)$, unless some centre-subspace-type patterns might appear due to nonlinearities involved. In any case, we believe that (C.30) describe (at least, a.a.) of generic formations of “multiple zeros” of solutions. In other words, for small $n > 0$, multiple zeros of the $\operatorname{Re} u(x, t)$ and/or $\operatorname{Im} u(x, t)$ as $(x, t) \rightarrow (0, 0^-)$ are created by a self-focusing of zero surfaces of the corresponding profiles $f(y)$ given in (C.28), which comprises a proper linear combination of the generalized Hermite polynomials (4.16). Indeed, this study requires further extensions, however, a complete and fully rigorous answer seems cannot be achieved.

With such an “optimistic point”, we end up the list of possible applications of our refined scattering linear spectral $\{\mathbb{B}, \mathbb{B}^*\}$ -theory for $2m$ th-order rescaled Schrödinger operators.

APPENDIX D. EIGENFUNCTION EXPANSIONS AND LITTLE HILBERT SPACES

Given below further developing of spectral theory is not necessary for our main applications concerning classification of all the global and blow-up asymptotics, which follow from the corresponding spectral decompositions of semigroup representation of solutions. However, we think that these accompanying results are interesting and actually allow to extend the results to wider classes of solutions.

Recall that the main difficulty with a proper definition of the operator \mathbb{B} (3.30) by its spectral decomposition (3.29) was associated with the fact that the necessary for us eigenfunctions $\{\psi_{\beta}\}$ were *extended*, i.e., did not belong to the present domain. In other words, these eigenfunctions, which inevitably appeared in eigenfunctions expansions (3.28) of rather “good” solutions of Schrödinger equations, were much “worse” than the solutions themselves. To get rid of such a controversy and to restore the true meaning of the operator pair $\{\mathbb{B}, \mathbb{B}^*\}$ (instead of its restriction $\{\mathbb{B}, \mathbb{B}^*\}$), we first introduce new *extended spaces of closures*.

D.1. Subspace where Φ is closed. Given the complete subset $\Phi = \{\psi_\beta\}$ for the non self-adjoint operator \mathbf{B} , we define the linear subspace \tilde{L}_ρ^2 of eigenfunction expansions,

$$(D.1) \quad v \in \tilde{L}_\rho^2 \quad \text{iff} \quad v = \sum c_\beta \psi_\beta \quad \text{with convergence in } L_\rho^2(\mathbb{R}^N),$$

as the closure of the subset of finite sums

$$(D.2) \quad \left\{ \sum_{|\beta| \leq K} c_\beta \psi_\beta, K \geq 0 \right\}$$

in the L_ρ^2 -norm.

For clarifying such a space, we now derive better estimates to see which $\{c_\beta\}$ satisfy (D.1). Namely, we will use the equality (3.17) for the key exponent in the exponential representation (2.11) of the asymptotics (which is responsible for the sharp estimate (3.16) of eigenfunctions ψ_β). Then similar to (3.25), but sharper, we then obtain for $l = |\beta| \gg 1$,

$$(D.3) \quad \int \rho |\psi_\beta|^2 \sim \left(\frac{l}{e}\right)^{-l} l^{\frac{2l(\alpha-1)}{\alpha}} \left[\frac{2(\alpha-1)}{\alpha l}\right]^{\frac{2l(\alpha-1)}{\alpha}} (2m)^{-\frac{2l}{2m-1}} = l^{-\frac{l(2-\alpha)}{\alpha}} \left[\frac{e^{2m-1}}{2m(me)^{(2m-1)/m}}\right]^{\frac{l}{2m-1}}.$$

Next, since

$$(D.4) \quad \|v\|_{L_\rho^2}^2 = \sum_{(\beta, \gamma)} c_\beta c_\gamma \int_{\mathbb{R}^N} \rho \psi_\beta \psi_\gamma,$$

(D.3) implies that:

$$(D.5) \quad v \in \tilde{L}_\rho^2 \quad \text{if} \quad c_\beta \text{ does not grow faster than } l^{2(\frac{2-\alpha}{\alpha}-\varepsilon)} \text{ for } l = |\beta| \gg 1,$$

where $\varepsilon > 0$ is any arbitrarily small constant. Of course, (D.3) defines more optimal and weaker inclusion conditions, but (D.5) clearly explains how this works.

D.2. Bi-orthonormality of the bases. Obviously, this is a principal issue for all the applications, where eigenfunction expansion techniques take part. As we have seen, the eigenfunctions expansions such as (3.32) introduce standard linear functionals $\langle w_0, \psi_\beta^* \rangle$, which are well defined for all functions $w_0 \in L_{\rho^*}^2(\mathbb{R}^N)$, so that, as usual, ψ_β^* is an element of the adjoint space $L_\rho^2(\mathbb{R}^N)$, with $\rho = \frac{1}{\rho^*}$, as customary.

As the next step, according to our construction above, we have to define some generalized *extended linear functionals* from the adjoint space \tilde{L}_ρ^{2*} . On one hand, this would correspond to a standard procedure of extension of such continuous uniformly convex functionals by the Hahn–Banach classic theorem in linear normed spaces, [51]. As we have seen earlier, those linear functionals are well defined according to (3.22) in $L_{\rho^*}^2$ with the standard (not in any *v.p.* or a canonical regularized, etc., sense) definition of the integrals.

On the other hand, such extended linear functionals cannot be understood in a standard sense, so that we refer to them as to *generalized* ones. It seems, a full proper definition of such extended linear functionals in a usual functional framework will require a deeper analysis of the actual functional spaces and metric/topologies involved, which will essentially decline us from main PDE applications [Especially, since in some of the applications, we do not and even cannot pretend to be mathematically rigorous.]

Therefore, in other words, for any $v \in \tilde{L}_\rho^2$, we define *extended linear functionals* for any β as:

$$(D.6) \quad \langle v, \psi_\beta^* \rangle_* \equiv \langle v, \bar{\psi}_\beta^* \rangle \quad \text{denotes the expansion coefficient } c_\beta \text{ of } v \text{ in (D.1).}$$

In view of the performed construction of the space \tilde{L}_ρ^2 via closure of finite sums (D.2), it is not difficult to see that such generalized continuous linear functionals are defined uniquely (in view of the density of finite sums (D.2)).

Overall, in the sense of (D.6), the standard bi-orthonormality of the bases $\{\psi_\beta\}$ and $\{\psi_\beta^*\}$ becomes trivial:

$$(D.7) \quad \langle \psi_\beta, \psi_\gamma^* \rangle_* \equiv \langle \psi_\beta, \bar{\psi}_\gamma^* \rangle = \delta_{\beta\gamma} \quad \text{for any } \beta \text{ and } \gamma,$$

where $\langle \cdot, \cdot \rangle$ is the usual duality product in $L^2(\mathbb{R}^N)$ and $\delta_{\beta\gamma}$ is the Kronecker delta.

Similarly, using the subset $\Phi^* = \{\psi_\beta^*\}$ of the generalized Hermite polynomials (4.16), we are obliged to define the corresponding subspace $\tilde{L}_{\rho,*}^2$ of eigenfunction expansions, and eventually treat similarly the adjoint extended linear functionals $\langle w, \psi_\beta \rangle_*$ for any $w \in \tilde{L}_{\rho,*}^2$.

In writing (D.7), we use the standard L^2 -metric, which is convenient to see the normalization since ψ_β is essentially the D^β derivative and ψ_β^* is a polynomial, so that, including normalization factors yields

$$(D.8) \quad \langle \psi_\beta, \bar{\psi}_\beta^* \rangle = \frac{(-1)^{|\beta|}}{\beta!} \int_{\mathbb{R}^N} D^\beta F(y) (y^\beta + \dots) dy = 1 \quad \text{for any multiindex } \beta$$

in the sense of formal integration by parts. Therefore, the integral itself is not of a standard meaning, but can be treated in an involved “*v.p.*-like sense”, which is difficult to clarify, and we do not feel any actual necessity to do this. A similar formalism exists for the whole eigenfunction set occurred in (D.7). Thus, we will use (D.7) in the eigenfunction expansions to follow, bearing in mind its actual meaning specified above in (D.6).

D.3. Little Hilbert and Sobolev spaces. It is convenient to introduce a *little* Hilbert space l_ρ^2 of functions $v = \sum a_\beta \psi_\beta \in \tilde{L}_\rho^2$ with coefficients satisfying

$$(D.9) \quad \sum |a_\beta|^2 < \infty,$$

where the scalar product and the induced norm are given by

$$(D.10) \quad (v, w)_0 = \sum a_\beta \bar{c}_\beta, \quad w = \sum c_\beta \psi_\beta \in l_\rho^2 \quad \text{and} \quad \|v\|_0^2 = (v, v)_0.$$

Therefore, Φ is now treated as a Riesz basis in $\tilde{L}_\rho^2(\mathbb{R}^N)$, [9, 32]. We next define a little Sobolev space h_ρ^{2m} of functions $v \in l_\rho^2$ such that

$$\sum |\lambda_\beta c_\beta|^2 < \infty.$$

The scalar product and the induced norm in h_ρ^{2m} are

$$(D.11) \quad (v, w)_1 = (v, w)_0 + (\mathbf{B}v, \mathbf{B}w)_0, \quad \|v\|_1^2 = (v, v)_1 \equiv \sum (1 + |\lambda_\beta|^2) |c_\beta|^2,$$

where our bounded operator $\mathbf{B} : h_\rho^{2m} \rightarrow l_\rho^2$ has the meaning $\mathbf{B} : \{c_\beta\} \rightarrow \{\lambda_\beta c_\beta\}$. This norm is equivalent to the graph norm induced by the positive operator $(-\mathbf{B} + aI)$ with $a > 0$. Then h_ρ^{2m} is the domain of \mathbf{B} in l_ρ^2 , and, by Sobolev’s embedding theorem,

$$(D.12) \quad h_\rho^{2m} \subset l_\rho^2 \quad \text{compactly,}$$

which follows from the criterion of compactness in l^p , [51].

D.4. **Basic properties in l_ρ^2 .** Firstly, it follows that \mathbf{B} is *self-adjoint* (symmetric) in l_ρ^2 ,

$$(D.13) \quad (\mathbf{B}v, w)_0 = (v, \mathbf{B}w)_0 \quad \text{for all } v, w \in h_\rho^{2m}.$$

Secondly, we state some other straightforward consequences.

Proposition D.1. (i) *The Hilbert space l_ρ^2 is a dense subspace of \tilde{L}_ρ^2 in $L_\rho^2(\mathbb{R}^N)$;*
(ii) *$\Phi = \{\psi_\beta\}$ is complete and closed in l_ρ^2 in the topology of $L_\rho^2(\mathbb{R}^N)$;*
(iii) *the resolvent $(\mathbf{B} - \lambda I)^{-1}$ for $\lambda \notin \sigma(\mathbf{B})$ is compact in l_ρ^2 ; and*
(iv) *\mathbf{B} is sectorial in l_ρ^2 .*

Proof. (i) l_ρ^2 is separable and complete since the same is true for the isomorphic Hilbert space l^2 of sequences. Let us show that $l_\rho^2 \subseteq \tilde{L}_\rho^2$. For any $v \in l_\rho^2$,

$$\int \rho |v|^2 dy = \int |\sum a_\beta \sqrt{\rho} \psi_\beta|^2 dy \leq \sum |a_\beta|^2 \sum \int e^{-|y|^\alpha} \left| \frac{1}{\sqrt{\beta!}} D^\beta F(y) \right|^2 dy,$$

and by the same estimates as in (3.23), (3.24), we conclude that, for $l = |\beta| \gg 1$,

$$(D.14) \quad \int e^{-|y|^\alpha} \left| \frac{1}{\sqrt{\beta!}} D^\beta F(y) \right|^2 dy \leq l^{l(-\nu+\varepsilon)},$$

where $\nu = \frac{2-\alpha}{\alpha} > 0$ and $\varepsilon > 0$ can be an arbitrarily small constant. Therefore, $l_\rho^2 \subset \tilde{L}_\rho^2$. Concerning the density of l_ρ^2 , we note that given a $v = \sum a_\beta \psi_\beta \in \tilde{L}_\rho^2$, the sequence of truncations $\{\sum_{|\beta| \leq K} a_\beta \psi_\beta, K = 1, 2, \dots\} \subset l_\rho^2$ converges to v in the topology of $L_\rho^2(\mathbb{R}^N)$ as $K \rightarrow \infty$ by completeness and closure of $\{\psi_\beta\}$.

(ii) Since Φ is orthonormal in l_ρ^2 , it follows that the only element orthogonal to $\{\psi_\beta\}$ is 0, and hence completeness of $\{\psi_\beta\}$ in l_ρ^2 follows from the Riesz–Fischer theorem. It is closed as an orthonormal subset in a separable Hilbert space [51].

(iii) For any $v = \sum a_\beta \psi_\beta \in l_\rho^2$ from the unit ball T_1 in l_ρ^2 with $\sum |a_\beta|^2 \leq 1$,

$$(D.15) \quad (\mathbf{B} - \lambda I)^{-1}v = \sum b_\beta \psi_\beta, \quad \text{where} \quad b_\beta = \frac{a_\beta}{\lambda_\beta - \lambda} = -\frac{a_\beta}{|\beta|/2m + \lambda} = -\frac{2ma_\beta}{|\beta|} \left[1 + O\left(\frac{1}{|\beta|}\right)\right] \quad \text{for } |\beta| \gg 1.$$

Therefore, for any $\varepsilon > 0$, there exists $K = K(\varepsilon) > 0$ such that for any $v \in T_1$,

$$\sum_{|\beta| \geq K} |b_\beta|^2 \leq \frac{4m^2}{K^2} \sum |a_\beta|^2 \leq \frac{4m^2}{K^2} < \varepsilon.$$

By the compactness criterion in l^2 [51], $(\mathbf{B} - \lambda I)^{-1}$ maps T_1 onto a compact subset in l_ρ^2 .

(iv) Recall that $(\mathbf{B} - \lambda I)^{-1}$ is a meromorphic function having a pole $\sim \frac{1}{\lambda}$ as $\lambda \rightarrow 0$ since $\lambda_0 = 0$ has multiplicity one [32]. We then need an extra estimate on the resolvent, which is easy to get in l_ρ^2 (one can check that it is not that easy in the big space L_ρ^2). In the sector $\Omega_\theta = \{\lambda \in \mathbb{C} : \lambda \neq 0, |\arg \lambda| < \frac{\pi}{2} + \theta\}$ with a $\theta \in (0, \frac{\pi}{2})$, for any $v = \sum a_\beta \psi_\beta \in l_\rho^2$, we apply (D.15) by using the fact that $\frac{1}{|\lambda_\beta - \lambda|} \leq \frac{1}{|\lambda|} \sin \theta$ in Ω_θ to get

$$\|(\mathbf{B} - \lambda I)^{-1}v\|_0 = \left(\sum |a_\beta|^2 \frac{1}{|\lambda_\beta - \lambda|^2}\right)^{\frac{1}{2}} \leq \frac{1}{\sin \theta} \frac{1}{|\lambda|} \|v\|_0.$$

Since \mathbf{B} is closed and densely defined, it is a sectorial operator in l_ρ^2 , see [22]. \square

Similarly to (D.1), for the adjoint operator \mathbf{B}^* we define the subspace $\tilde{L}_{\rho,*}^2 \subset L_{\rho}^2(\mathbb{R}^N)$, where the eigenfunction subset Φ^* is closed, $l_{\rho,*}^2$, $h_{\rho,*}^{2m}$, etc., and next continue develop similar theory of self-adjointness and other properties

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